Monetary Economics: Problem Set #2

Solutions

This problem set is marked out of 100 points. The weight given to each part is indicated below. Please contact me asap if you have any questions.

1. **Inflation targeting with noisy data.** Consider a new Keynesian model with output gap and inflation dynamics governed by

$$\tilde{y}_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n) + \mathbb{E}_t\{\tilde{y}_{t+1}\}$$

and

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa \tilde{y}_t$$

where all variables have their usual meanings. The natural rate of interest follows an exogenous AR(1) process

$$r_{t+1}^n - \rho = \rho_r(r_t^n - \rho) + \epsilon_{t+1}$$

with $0 \leq \rho_r < 1$ and where $\{\epsilon_t\}$ is IID white noise with mean zero.

Now suppose that inflation is observed with *measurement error* so that the central bank sees only a noisy signal of actual inflation

$$\pi_t^0 = \pi_t + \xi_t$$

where π_t^0 denotes the observed inflation rate, π_t the actual inflation rate, and where $\{\xi_t\}$ is IID white noise with mean zero. Assume the central bank follows the feedback rule

$$i_t = \rho + \phi_\pi \pi_t^0 \tag{1}$$

in terms of observed inflation.

(a) Use the method of undetermined coefficients to solve for the equilibrium processes for inflation and the output gap under the interest rate rule. (20 points)

(b) Describe the behavior of inflation, the output gap, and the nominal interest rate when $\phi_\pi \to \infty$. Give economic intuition for your answers. (15 points)

To simplify the algebra, assume for the rest of this question that $\rho_r = 0$.

(c) Determine the value of the feedback coefficient ϕ_π that minimises the variance of actual inflation. Give economic intuition for your answer. (15 points)

Solutions:

(a) Let $\hat{r}_t^n \equiv r_t^n - \rho$ and substitute in the interest rate rule so that we can reduce the system to

$$\tilde{y}_t = -\frac{1}{\sigma}(\phi_\pi(\pi_t + \xi_t) - \mathbb{E}_t\{\pi_{t+1}\} - \hat{r}_t^n) + \mathbb{E}_t\{\tilde{y}_{t+1}\}$$

and

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa\tilde{y}$$

Now guess solutions of the form

$$\tilde{y}_t = \varphi_{yr}\hat{r}_t^n + \varphi_{y\xi}\xi_t, \qquad \text{and} \qquad \pi_t = \varphi_{\pi r}\hat{r}_t^n + \varphi_{\pi\xi}\xi_t$$

for some as-yet-unknown coefficients $\varphi_{yr}, \varphi_{y\xi}, \varphi_{\pi r}, \varphi_{\pi\xi}$ to be determined. Since the natural rate follows an AR(1) with coefficient ρ_r and ξ_t is IID white noise with mean zero, these solutions imply the conditional expectations

$$\mathbb{E}_t\{\tilde{y}_{t+1}\} = \varphi_{yr}\rho_r\hat{r}_t^n, \qquad \text{and} \qquad \mathbb{E}_t\{\pi_{t+1}\} = \varphi_{\pi r}\rho_r\hat{r}_t^n$$

So we can write the system of two equations in terms of the four unknown coefficients and the shocks $\hat{r}_t^n, \xi_t$, specifically

$$\varphi_{yr}\hat{r}_t^n + \varphi_{y\xi}\xi_t = -\frac{1}{\sigma}(\phi_\pi(\varphi_{\pi r}\hat{r}_t^n + \varphi_{\pi\xi}\xi_t + \xi_t) - \varphi_{\pi r}\rho_r\hat{r}_t^n - \hat{r}_t^n) + \varphi_{yr}\rho_r\hat{r}_t^n$$

and

$$\varphi_{\pi r}\hat{r}_t^n + \varphi_{\pi\xi}\xi_t = \beta\varphi_{\pi r}\rho_r\hat{r}_t^n + \kappa(\varphi_{yr}\hat{r}_t^n + \varphi_{y\xi}\xi_t)$$

Collecting terms in the shocks we have

$$0 = \left[\varphi_{yr} + \frac{1}{\sigma}(\phi_\pi\varphi_{\pi r} - \varphi_{\pi r}\rho_r - 1) - \varphi_{yr}\rho_r\right]\hat{r}_t^n + \left[\varphi_{y\xi} + \frac{1}{\sigma}\phi_\pi(\varphi_{\pi\xi} + 1)\right]\xi_t$$

and

$$0 = \left[\varphi_{\pi r} - \beta\varphi_{\pi r}\rho_r - \kappa\varphi_{yr}\right]\hat{r}_t^n + \left[\varphi_{\pi\xi} - \kappa\varphi_{y\xi}\right]\xi_t$$

Since these conditions have to hold for *any* realizations of the shock $\hat{r}_t^n, \xi_t$ the terms in square brackets must each be zero so that we are left with four conditions in the four unknown coefficients. Doing the algebra gives the solutions

$$\varphi_{yr} = \frac{1 - \beta\rho_r}{\sigma(1-\beta\rho_r)(1-\rho_r) + \kappa(\phi_\pi - \rho_r)} > 0$$
$$\varphi_{\pi r} = \frac{\kappa}{\sigma(1-\beta\rho_r)(1-\rho_r) + \kappa(\phi_\pi - \rho_r)} > 0$$

(assuming $\phi_\pi > 1$, as usual) and

$$\varphi_{y\xi} = -\frac{\phi_\pi}{\sigma + \kappa\phi_\pi} < 0$$
$$\varphi_{\pi\xi} = -\frac{\kappa\phi_\pi}{\sigma + \kappa\phi_\pi} < 0$$

(b) When $\phi_\pi \to \infty$, these coefficients simplify to

$$\varphi_{yr} = 0$$
$$\varphi_{\pi r} = 0$$

and, using l'Hopital's rule, also

$$\varphi_{y\xi} = -\frac{1}{\kappa} < 0$$
$$\varphi_{\pi\xi} = -1$$

Therefore in this limit, equilibrium inflation is

$$\pi_t = \varphi_{\pi r}\hat{r}_t^n + \varphi_{\pi\xi}\xi_t = 0\,\hat{r}_t^n - 1\,\xi_t = -\xi_t$$

hence equilibrium inflation is perfectly negatively correlated with the measurement error. This is because observed the monetary authority is very reactive and inflation is $\pi_t^0 \equiv \pi_t + \xi_t = 0$ in this limit (i.e., the central bank is successfully keeping observed inflation at zero). Similarly, in this limit the equilibrium output gap is

$$\tilde{y}_t = \varphi_{yr}\hat{r}_t^n + \varphi_{y\xi}\xi_t = 0\,\hat{r}_t^n - \frac{1}{\kappa}\,\xi_t = -\frac{1}{\kappa}\xi_t$$

Calculating the equilibrium nominal interest rate is slightly more tricky. From the interest rule evaluated at the equilibrium observed inflation rate

$$i_t = \rho + \phi_\pi \pi_t^0 = \rho + \phi_\pi(\pi_t + \xi_t)$$

Plugging in for equilibrium inflation

$$\begin{aligned} i_t &= \rho + \phi_\pi(\pi_t + \xi_t) \\ &= \rho + \phi_\pi\varphi_{\pi r}\hat{r}_t^n + \phi_\pi(\varphi_{\pi\xi} + 1)\xi_t \end{aligned}$$

Now the coefficient on the natural real rate is

$$\phi_\pi\varphi_{\pi r} = \phi_\pi \frac{\kappa}{\sigma(1-\beta\rho_r)(1-\rho_r) + \kappa(\phi_\pi - \rho_r)} \to 1 \quad \text{as} \quad \phi_\pi \to \infty$$

And the coefficient on the measurement error is

$$\phi_\pi(\varphi_{\pi\xi} + 1) = \phi_\pi\left(1 - \frac{\kappa\phi_\pi}{\sigma + \kappa\phi_\pi}\right) = \phi_\pi\frac{\sigma}{\sigma + \kappa\phi_\pi} \to \frac{\sigma}{\kappa} \quad \text{as} \quad \phi_\pi \to \infty$$

Therefore in this limit, the nominal interest rate is

$$i_t = \rho + \hat{r}_t^n + \frac{\sigma}{\kappa}\xi_t = \rho + (r_t^n - \rho) + \frac{\sigma}{\kappa}\xi_t = r_t^n + \frac{\sigma}{\kappa}\xi_t$$

In the absence of measurement error, this would just be the natural real rate as usual.

(c) When the natural rate follows an AR(1), as in parts (a)-(b) above, the variance of inflation is

$$\text{Var}[\pi_t] = \left(\frac{\kappa}{\sigma(1-\beta\rho_r)(1-\rho_r) + \kappa(\phi_\pi - \rho_r)}\right)^2 \text{Var}[\hat{r}_t^n] + \left(\frac{\kappa\phi_\pi}{\sigma + \kappa\phi_\pi}\right)^2 \text{Var}[\xi_t]$$

(using the fact that the shocks are independent). Trying to mimimize this leads to a bit of a mess. But you were told to make life easier by setting $\rho_r = 0$, so that the natural real rate is IID over time, in which case we have the simpler expression

$$\text{Var}[\pi_t] = \left(\frac{\kappa}{\sigma + \kappa\phi_\pi}\right)^2 \text{Var}[\hat{r}_t^n] + \left(\frac{\kappa\phi_\pi}{\sigma + \kappa\phi_\pi}\right)^2 \text{Var}[\xi_t]$$

The first order condition for a minimum is

$$\frac{d}{d\phi_\pi}\text{Var}[\pi_t] = -\frac{2\kappa^3}{(\sigma + k\phi_\pi)^3}\text{Var}[\hat{r}_t^n] + \frac{2\kappa^2\sigma\phi_\pi}{(\sigma + \kappa\phi_\pi)^3}\text{Var}[\xi_t] = 0$$

Solving for ϕ_π gives

$$\phi_\pi = \frac{\kappa}{\sigma}\frac{\text{Var}[\hat{r}_t^n]}{\text{Var}[\xi_t]}$$

and you can check the second order condition is satisfied at this point, so this is a global minimum. Intuitively, the more noisy is the observation of inflation (the bigger is $\text{Var}[\xi_t]$), the smaller the weight the central bank should attach to inflation (if its objective is to minimize the variance of inflation).

2. **Monetary policy and the effects of productivity shocks.** Consider a new Keynesian model with equilibrium conditions

$$y_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho) + \mathbb{E}_t\{y_{t+1}\} \tag{2}$$

and

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa(y_t - y_t^n) \tag{3}$$

where all variables have their usual meanings. Monetary policy is given by the feedback rule

$$i_t = \rho + \phi_\pi \pi_t$$

where $\phi_\pi > 1$. The production function (in logs) is

$$y_t = a_t + n_t$$

where a_t is an exogenous labor productivity process that follows an AR(1) process

$$a_{t+1} = \rho_a a_t + \epsilon_{t+1}$$

with $0 \leq \rho_a < 1$ and where $\{\epsilon_t\}$ is IID white noise with mean zero. Natural output is proportional to productivity

$$y_t^n = \psi_y a_t$$

where $\psi_y > 0$.

(a) Describe in words the economic interpretation of equations (2) and (3). (10 points)

(b) Use the method of undetermined coefficients to solve for the equilibrium response of output, employment, and inflation to a productivity shock. (20 points)

(c) Describe how these responses depend on the values of the parameters ϕ_π and κ. What happens when $\phi_\pi \to \infty$? What happens as the degree of price rigidities changes? Provide economic intuition for your answers. (10 points)

(d) Discuss with as much detail as you can the joint response of employment and output to a productivity shock and discuss the implications for assessing the role of productivity shocks as a source of business cycle fluctuations in this model. (10 points)

Hint: you may want to skim Gali's 1999 paper "Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?" (posted on the LMS).

Solutions:

(a) Equation (2) is the standard (log-linear) intertemporal consumption Euler equation *plus* a simple goods market clearing condition of the form $c_t = y_t$. Equation (3) is the new Keynesian Phillips curve. This starts with an imperfectly competitive firm's optimal price-setting behavior subject to a Calvo-style price-setting rigidity. This is then log-linearized and the terms reflecting real marginal cost are eliminated using the household's labor supply condition and approximate resource constraint so that it can be written in terms of the output gap $y_t - y_t^n$ where natural output y_t^n is the level of output that would obtain in the same model but with perfectly flexible price-setting.

(b) Eliminating the nominal interest rate using the monetary policy rule $i_t = \rho + \phi_\pi \pi_t$ and eliminating natural output using productivity $y_t^n = \varphi_y a_t$ gives the system of two equations

$$y_t = -\frac{1}{\sigma}\left(\phi_\pi \pi_t - \mathbb{E}_t\{\pi_{t+1}\}\right) + \mathbb{E}_t\{y_{t+1}\}$$

and

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa(y_t - \varphi_y a_t)$$

We now guess solutions of the form

$$y_t = \varphi_{ya} a_t, \qquad \text{and} \qquad \pi_t = \varphi_{\pi a} a_t$$

for some as-yet-unknown coefficients $\varphi_{ya}, \varphi_{\pi a}$ to be determined. Since productivity follows an AR(1) with coefficient ρ_a, these solutions imply the conditional expectations

$$\mathbb{E}_t\{y_{t+1}\} = \varphi_{ya}\rho_a a_t, \qquad \text{and} \qquad \mathbb{E}_t\{\pi_{t+1}\} = \varphi_{\pi a}\rho_a a_t$$

So we can write the system of two equations in terms of these two unknown coefficients and the shock a_t, specifically

$$\varphi_{ya} a_t = -\frac{1}{\sigma}\left(\phi_\pi \varphi_{\pi a} - \varphi_{\pi a}\rho_a\right) a_t + \varphi_{ya}\rho_a a_t$$

and

$$\varphi_{\pi a} a_t = \beta \varphi_{\pi a}\rho_a a_t + \kappa(\varphi_{ya} - \varphi_y) a_t$$

Since these conditions have to hold for *any* realization of the shock a_t we are left with two conditions in the two unknown coefficients, that is

$$\varphi_{ya} = -\frac{1}{\sigma}\left(\phi_\pi \varphi_{\pi a} - \varphi_{\pi a}\rho_a\right) + \varphi_{ya}\rho_a$$

and

$$\varphi_{\pi a} = \beta\varphi_{\pi a}\rho_a + \kappa(\varphi_{ya} - \varphi_y)$$

Solving these two equations in the two unknown coefficients gives

$$\varphi_{ya} = \frac{\kappa(\phi_\pi - \rho_a)}{\sigma(1-\rho_a)(1-\beta\rho_a) + \kappa(\phi_\pi - \rho_a)}\varphi_y > 0$$

and

$$\varphi_{\pi a} = -\frac{\kappa\sigma(1-\rho_a)}{\sigma(1-\rho_a)(1-\beta\rho_a) + \kappa(\phi_\pi - \rho_a)}\varphi_y < 0$$

To sign these unambiguously, we've used the assumption made in the question that $\phi_\pi > 1$ so that $\phi_\pi - \rho_a > 0$. Therefore the equilibrium response of output and inflation to a productivity shock a_t are simply

$$\frac{\partial y_t}{\partial a_t} = \varphi_{ya} > 0$$

and

$$\frac{\partial \pi_t}{\partial a_t} = \varphi_{\pi a} < 0$$

Now from the approximate resource constraint (ignoring price dispersion, as usual) $y_t = a_t + n_t$ so in equilibrium

$$\frac{\partial n_t}{\partial a_t} = \varphi_{ya} - 1$$

This is not unambiguously signed. Substituting in the formula for φ_{ya} we have

$$\frac{\partial n_t}{\partial a_t} > 0 \Leftrightarrow \frac{\kappa(\phi_\pi - \rho_a)}{\sigma(1-\rho_a)(1-\beta\rho_a) + \kappa(\phi_\pi - \rho_a)}\varphi_y > 1$$

or equivalently

$$\varphi_y > \frac{\sigma(1-\rho_a)(1-\beta\rho_a)}{\kappa(\phi_\pi - \rho_a)} + 1$$

Hence the sensitivity of natural output y^n to productivity a has to be sufficiently large in order for employment n to rise.

(c) The response of output to productivity, φ_{ya}, is increasing in both κ and ϕ_π (since $\phi_\pi > 1$, by assumption). The *magnitude* of the response of inflation to productivity, $|\varphi_{\pi a}|$, is also increasing in κ but is decreasing in ϕ_π. As $\phi_\pi \to \infty$, the response of inflation to productivity $\varphi_{\pi a} \to 0$ so that inflation is also driven to zero for all realizations of productivity a_t and so of course the variance of inflation is zero too. Similarly, when $\phi_\pi \to \infty$ the response of output $\varphi_{ya} \to \varphi_y$, i.e., the underlying response of natural output to productivity. Since the output *gap* is $\tilde{y}_t \equiv y_t - y_t^n = (\varphi_{ya} - \varphi_y)a_t$, as $\varphi_{ya} \to \varphi_y$ the output gap is driven to zero for all realizations of productivity a_t so that the variance of the output gap is zero too.

The degree of price stickiness is captured by κ (it is strictly decreasing in the Calvo parameter θ – high θ (lots of price stickiness) means low κ – etc). As $\kappa \to 0$ we have $\varphi_{\pi a} \to 0$ and $\varphi_{ya} \to 0$ so that inflation is zero (lots of price stickiness!) and the output gap is $-\varphi_y a_t$. As $\kappa \to \infty$, we have $\varphi_{ya} \to \varphi_y$ (so no output gap) and $\varphi_{\pi a} \to -\varphi_y$ (inflation negatively correlated with productivity).

(d) The key property of the joint response of employment and output to a productivity shock is the *covariance* of employment and output. Since

$$y_t = \varphi_{ya} a_t$$

and

$$n_t = y_t - a_t = (\varphi_{ya} - 1)a_t$$

are both linear functions of a single underlying shock, the covariance between them is just

$$\text{Cov}[y_t, n_t] = \text{Cov}[\varphi_{ya} a_t, (\varphi_{ya} - 1)a_t] = \varphi_{ya}(\varphi_{ya} - 1)\text{Var}[a_t]$$

Since φ_{ya} and the variance $\text{Var}[a_t]$ are positive, the covariance is positive or negative depending on whether employment n_t rises or falls when productivity increases. For example, if $\varphi_{ya} > 1$ so that employment rises when productivity rises, then employment and output will have positive covariance (indeed a correlation coefficient of $+1$) while if $\varphi_{ya} < 1$ so that employment falls when productivity rises, then employment and output will have negative covariance (indeed a correlation coefficient of -1).

A real business cycle model, like the classical economy discussed in earlier lectures, usually has the implication that employment and output are positively correlated with each other and with productivity. Gali (AER, 1999 and subsequent papers) uses a VAR approach to argue that labor productivity and employment are negatively correlated in the data and that this should be taken as evidence against RBC models.