Advanced Macroeconomics

Lecture 4: growth theory and dynamic optimization, part three

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This class

- Stability of *systems* of difference equation
  - eigenvalues, eigenvectors etc
  - 'diagonalizing' systems of difference equations
  - implications for stability of linear dynamic systems

Recall scalar case

- Scalar linear difference equation

$$x_{t+1} = a x_t + b, \qquad x_0 \text{ given}$$

- If $a \neq 1$

$$x_t = \bar{x} + a^t (x_0 - \bar{x}), \qquad t \geq 0$$

with steady state

$$\bar{x} = (1 - a)^{-1} b$$

System of linear difference equations

- Now let's consider a *system* of linear difference equations

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

or in matrix notation

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{b}$$

- Analogous steady state

$$\bar{\boldsymbol{x}} = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$$

(supposing the inverse is well-defined, more on this soon)

Systems of linear difference equations

- Analogous solution

$$\boldsymbol{x}_t = \bar{\boldsymbol{x}} + \boldsymbol{A}^t(\boldsymbol{x}_0 - \bar{\boldsymbol{x}})$$

- Scalar dynamics characterized by behavior of a^t

- System dynamics characterized by behavior of $\boldsymbol{A}^t$

- But matrix power $\boldsymbol{A}^t$ is a complicated object. In general it is *not* the matrix of powers

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^t \neq \begin{pmatrix} a_{11}^t & a_{12}^t \\ a_{21}^t & a_{22}^t \end{pmatrix}$$

How then do we determine behavior of $\boldsymbol{A}^t$?

Uncoupled systems

- Consider *uncoupled* system

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

  Coefficient matrix $\boldsymbol{A}$ is *diagonal*, no feedback between components

- In this special case it *is* true that

$$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}^t = \begin{pmatrix} a_{11}^t & 0 \\ 0 & a_{22}^t \end{pmatrix}$$

- So, in this special case, the behavior of $\boldsymbol{A}^t$ simply determined by magnitudes of a_{11} and a_{22}

Diagonalizing a system

- Most systems of interest are coupled, matrix $\boldsymbol{A}$ not diagonal

- But large class of matrixes can be *diagonalized*. For these matrixes

  $$\boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V} = \boldsymbol{\Lambda}$$

  where $\boldsymbol{\Lambda}$ is a diagonal matrix with entries equal to the *eigenvalues* of $\boldsymbol{A}$ and $\boldsymbol{V}$ is a matrix which stacks the corresponding *eigenvectors* (more on these shortly)

- We then make the change of variables $\boldsymbol{z}_t \equiv \boldsymbol{V}^{-1}(\boldsymbol{x}_t - \bar{\boldsymbol{x}})$ and study

  $$\boldsymbol{V}\boldsymbol{z}_{t+1} = \boldsymbol{A}\boldsymbol{V}\boldsymbol{z}_t$$

  that is, the uncoupled system

  $$\boldsymbol{z}_{t+1} = \boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V}\boldsymbol{z}_t = \boldsymbol{\Lambda}\boldsymbol{z}_t$$

Diagonalizing a system

- Solving the uncoupled system

$$\boldsymbol{z}_t = \boldsymbol{\Lambda}^t \boldsymbol{z}_0$$

or in terms of the original coordinates

$$\boldsymbol{x}_t = \bar{\boldsymbol{x}} + \boldsymbol{V}\boldsymbol{z}_t = \bar{\boldsymbol{x}} + \boldsymbol{V}\boldsymbol{\Lambda}^t\boldsymbol{V}^{-1}(\boldsymbol{x}_0 - \bar{\boldsymbol{x}})$$

- These are just linear combinations of λ^t terms from diagonal of $\boldsymbol{\Lambda}^t$
- In short, eigenvalues λ of $\boldsymbol{A}$ determine stability of $\boldsymbol{x}_t$
- So what are these eigenvalues?

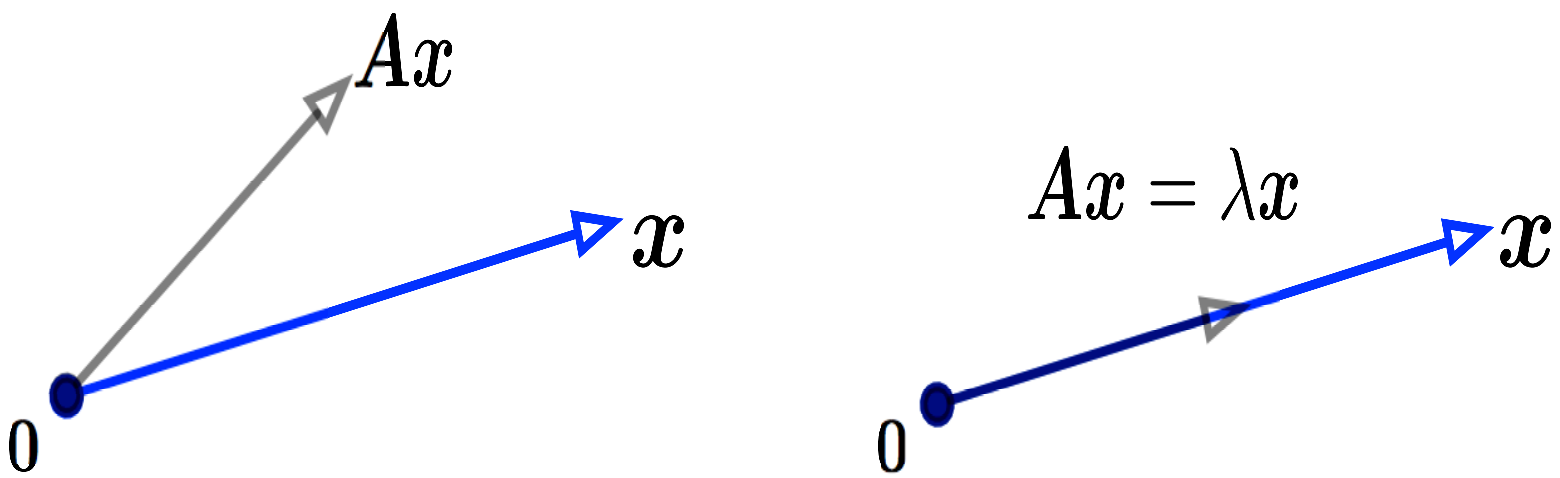
Eigenvalues and eigenvectors

If $\boldsymbol{A}$ is an $n \times n$ matrix, then a non-zero $n \times 1$ vector $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ if $\boldsymbol{A}\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

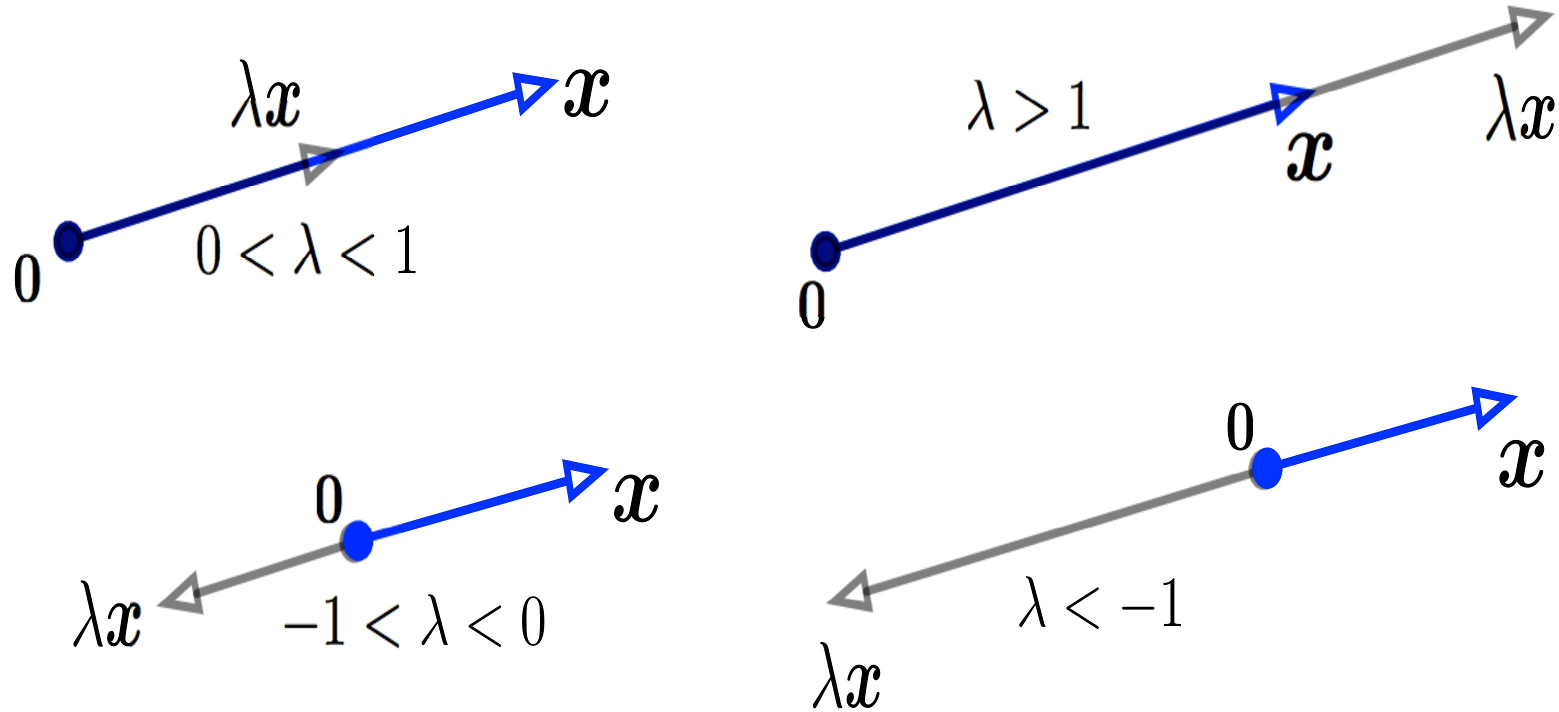
for some scalar λ. We then say λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is an eigenvector corresponding to λ.

Geometric interpretation



In general $\boldsymbol{Ax}$ is not proportional to $\boldsymbol{x}$. But if it is, then λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ corresponding to λ.

Magnitudes of eigenvalues



$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$

- So a scalar λ is an eigenvalue of a square matrix $\boldsymbol{A}$ iff

  $$\boldsymbol{M} \equiv \boldsymbol{A} - \lambda\boldsymbol{I}$$

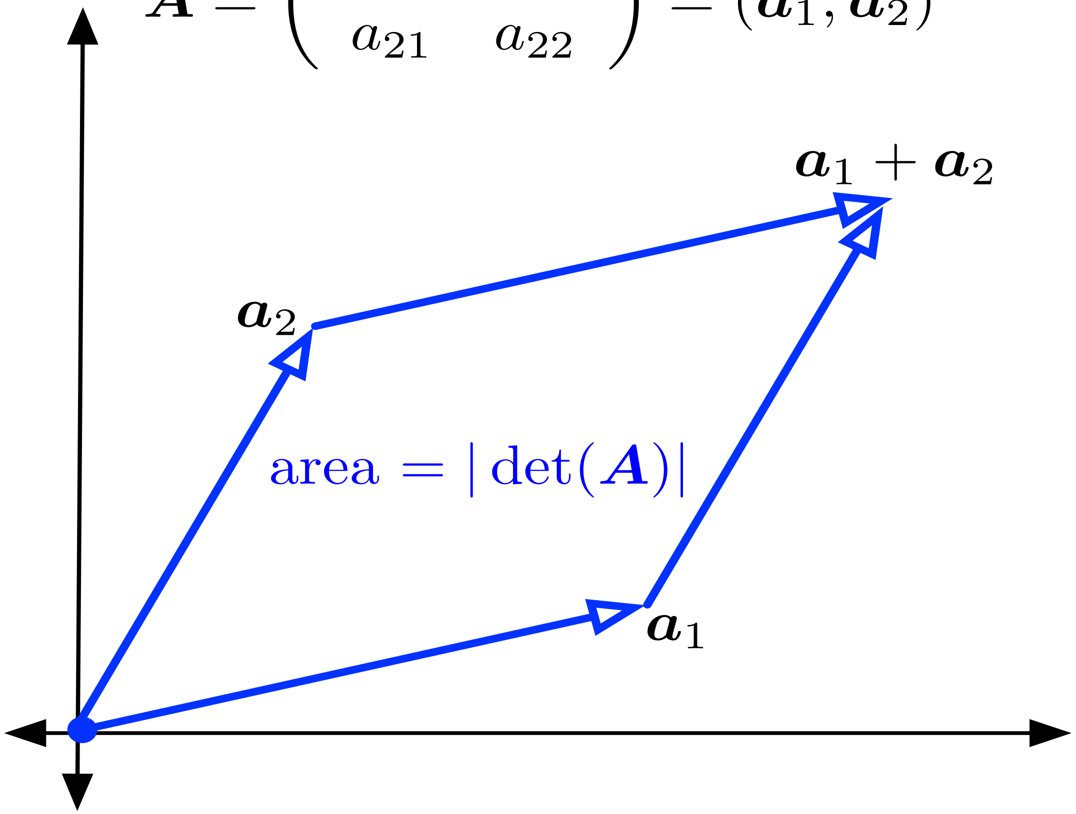
  is *singular*

  $\Leftrightarrow$ there are solutions to $\boldsymbol{M}\boldsymbol{x} = \mathbf{0}$ other than $\boldsymbol{x} = \mathbf{0}$
  $\Leftrightarrow$ the *determinant* of $\boldsymbol{M}$ is zero

- Consider scalar a. Let $m \equiv a - \lambda$. When does $mx = 0$ have solutions other than $x = 0$? When $m = 0$. When is $m = 0$? When $\lambda = a$. For scalar a, single eigenvalue equal to coefficient itself

Determinant: main idea

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$$



Absolute value of determinant of $\boldsymbol{A}$ equals area formed from columns $\boldsymbol{a}_1, \boldsymbol{a}_2$ of $\boldsymbol{A}$. Determinant equals zero if columns linearly dependent (in which case matrix is *singular*, parallelogram collapses to a line).

Finding eigenvalues

- The λ are the numbers that make $\boldsymbol{M} = \boldsymbol{A} - \lambda \boldsymbol{I}$ singular

- Matrix $\boldsymbol{M}$ singular when its determinant is zero. In 2-by-2 case

$$\det(\boldsymbol{M}) = \det \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = m_{11}m_{22} - m_{12}m_{21}$$

- Therefore

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

Characteristic polynomial

- So for a 2-by-2 $\boldsymbol{A}$, the eigenvalues λ solve a quadratic equation

$$p(\lambda) \equiv \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

(the '*characteristic polynomial*')

- Two roots. From the quadratic formula

$$\lambda_1, \lambda_2 = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

If $\boldsymbol{A}$ diagonal, roots are simply $\lambda_1 = a_{11}$ and $\lambda_2 = a_{22}$

- More generally nth order polynomial, n roots. Roots may be real or complex, repeated or distinct

- Repeated roots *may* lead to non-diagonalizable ('*defective*') matrices, i.e., have less than n linearly independent eigenvectors

Finding eigenvectors

- Suppose λ is an eigenvalue of $\boldsymbol{A}$ (from characteristic polynomial)

- Find *eigenvector* $\boldsymbol{x}$ associated with λ by solving

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = 0$$

- In 2-by-2 case

$$\begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  Here λ is fixed and we solve for $\boldsymbol{x}$

- Eigenvector not unique, if $\boldsymbol{x}$ is an eigenvector associated with λ then so is $c\boldsymbol{x}$ for any $c \neq 0$. Needs a normalization.

Implications for stability

- Recall

$$\boldsymbol{x}_t = \bar{\boldsymbol{x}} + \boldsymbol{V}\boldsymbol{z}_t = \bar{\boldsymbol{x}} + \boldsymbol{V}\boldsymbol{\Lambda}^t\boldsymbol{z}_0$$

- That is, linear combinations of eigenvalues of the form

$$x_{1,t} = \bar{x}_1 + v_{11}\lambda_1^t z_{1,0} + v_{12}\lambda_2^t z_{2,0}$$

$$x_{2,t} = \bar{x}_2 + v_{21}\lambda_1^t z_{1,0} + v_{22}\lambda_2^t z_{2,0}$$

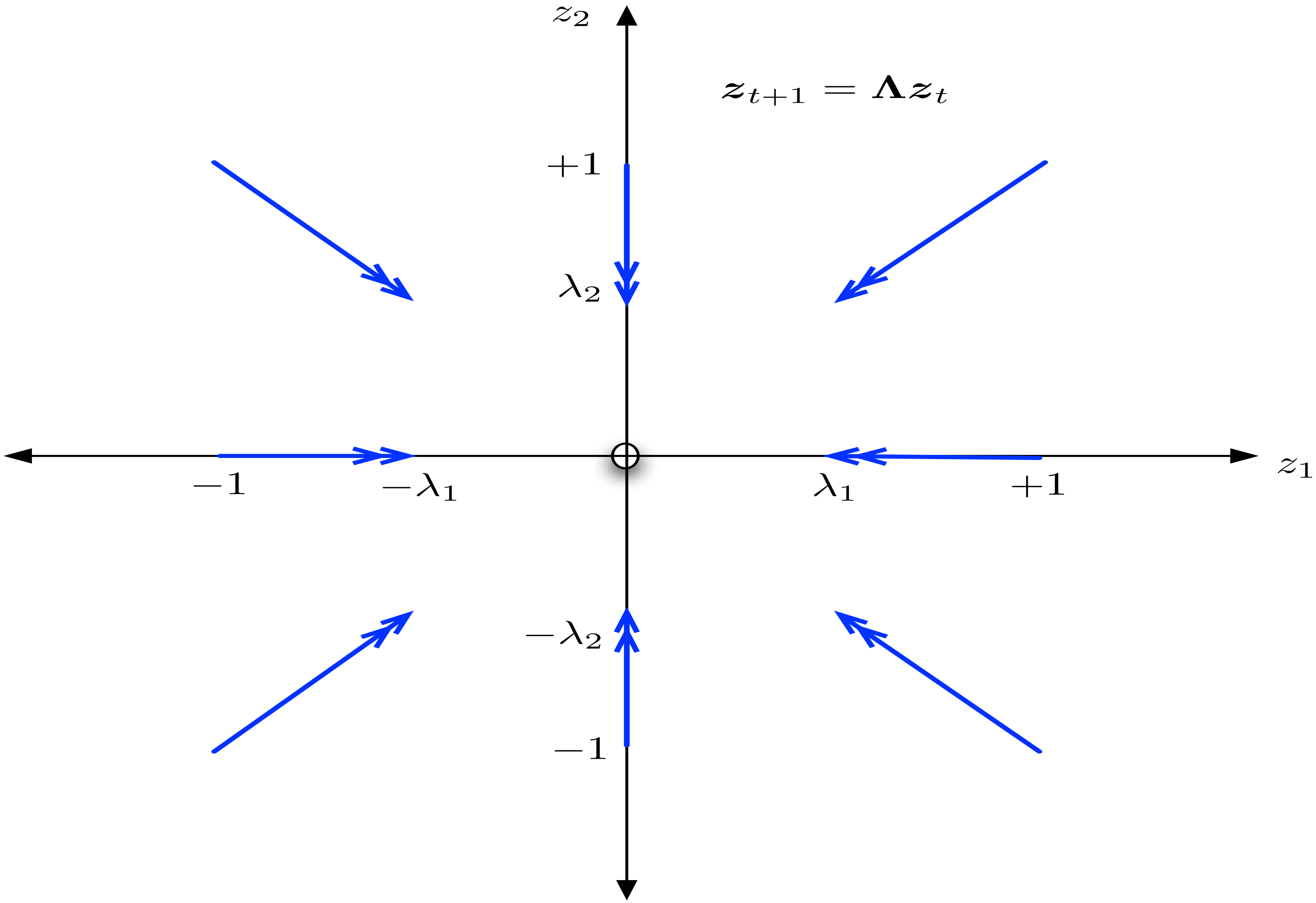
- Stable if all $|\lambda| < 1$, unstable otherwise. Note initial conditions

$$z_{1,0} = \frac{v_{22}(x_{1,0} - \bar{x}_1) - v_{12}(x_{2,0} - \bar{x}_2)}{v_{11}v_{22} - v_{12}v_{21}}$$

$$z_{2,0} = \frac{v_{11}(x_{2,0} - \bar{x}_2) - v_{21}(x_{1,0} - \bar{x}_1)}{v_{11}v_{22} - v_{12}v_{21}}$$

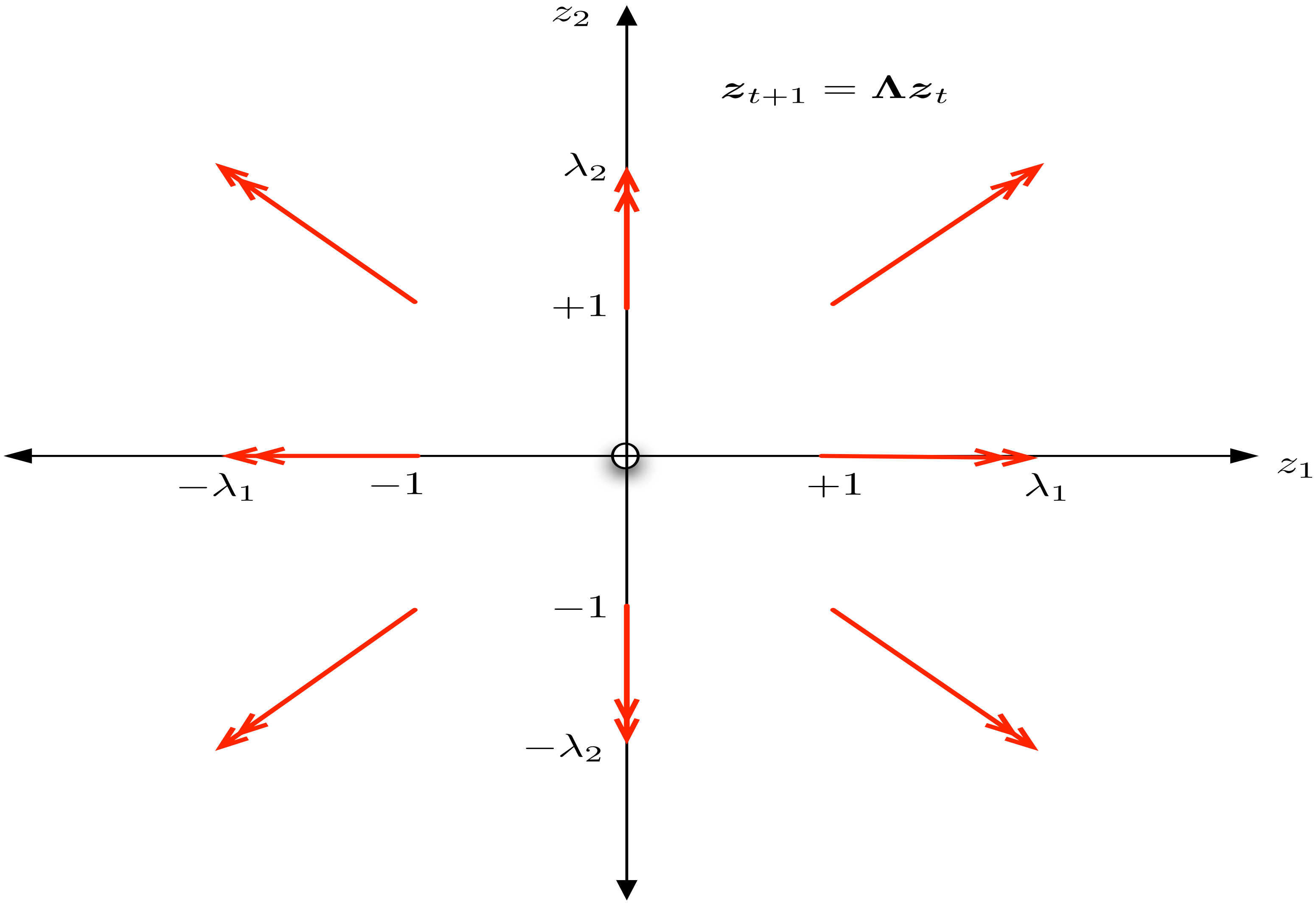
- An unstable λ dominates unless initial conditions are 'just right'.

Sink (all $|\lambda| < 1$)



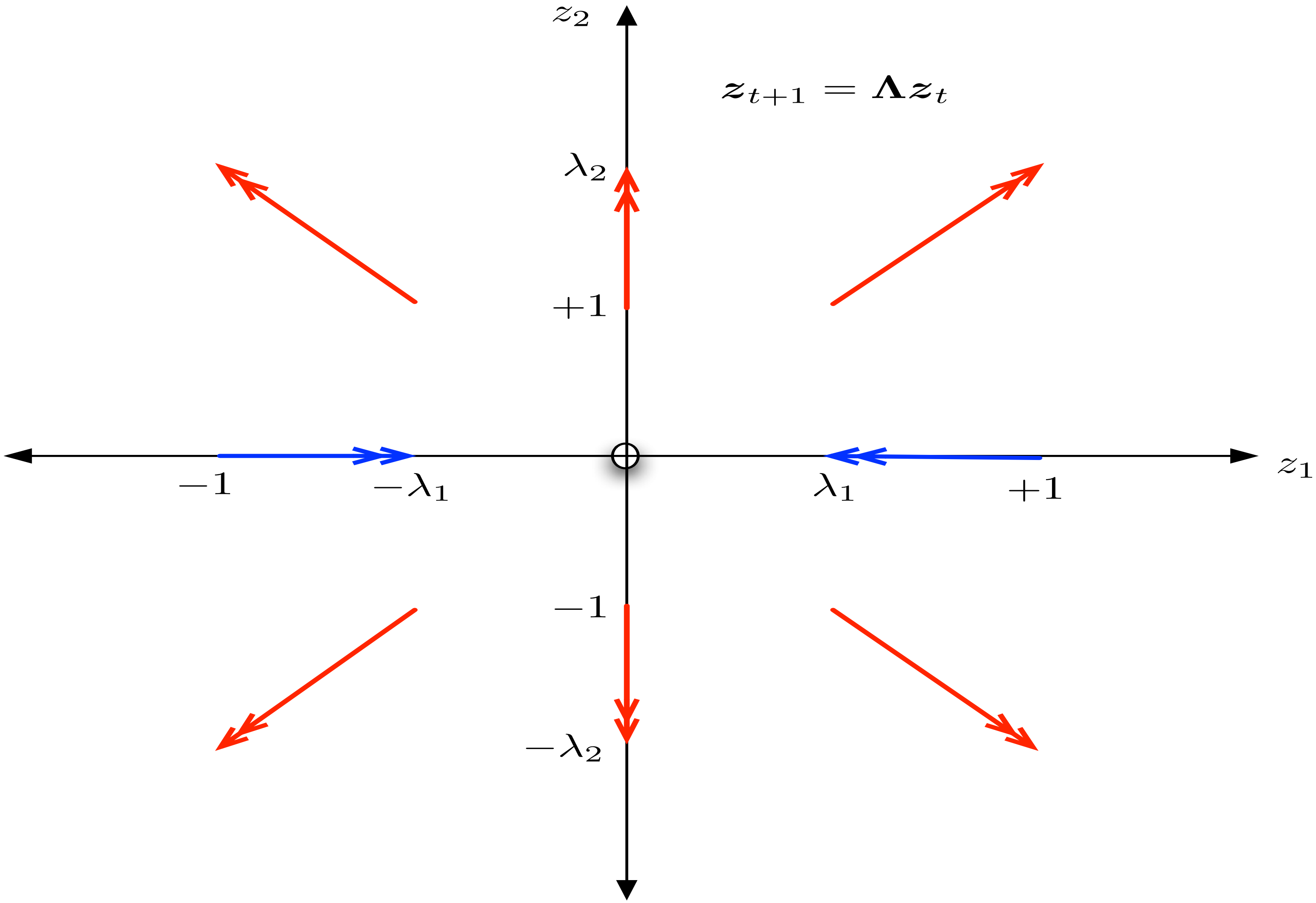
For any initial $\boldsymbol{z}_0$, system $\boldsymbol{z_t} = \boldsymbol{\Lambda}^t \boldsymbol{z}_0 \to \mathbf{0}$ (the origin) hence $\boldsymbol{x}_t \to \bar{\boldsymbol{x}}$.
Example shown here has both roots $\lambda_1, \lambda_2 \in (0, 1)$.

Source (all $|\lambda| > 1$)



For any initial $\boldsymbol{z}_0 \neq \mathbf{0}$, system $\boldsymbol{z_t} = \boldsymbol{\Lambda}^t \boldsymbol{z}_0$ diverges hence $\boldsymbol{x}_t$ diverges too.
Example shown here has both roots $\lambda_1, \lambda_2 > 1$.

Saddle (some $|\lambda| > 1$)



System $\boldsymbol{z_t} = \boldsymbol{\Lambda}^t \boldsymbol{z}_0$ diverges if *any* weight given to unstable roots.
Example here has $0 < \lambda_1 < 1 < \lambda_2$ and $\boldsymbol{z_t} = \boldsymbol{\Lambda}^t \boldsymbol{z}_0 \to \mathbf{0}$ iff $z_{2,0} = 0$.

Saddle (some $|\lambda| > 1$)

- Unstable eigenvalue dominates unless initial conditions 'just right'

- As in the last example, suppose

$$0 < \lambda_1 < 1 < \lambda_2$$

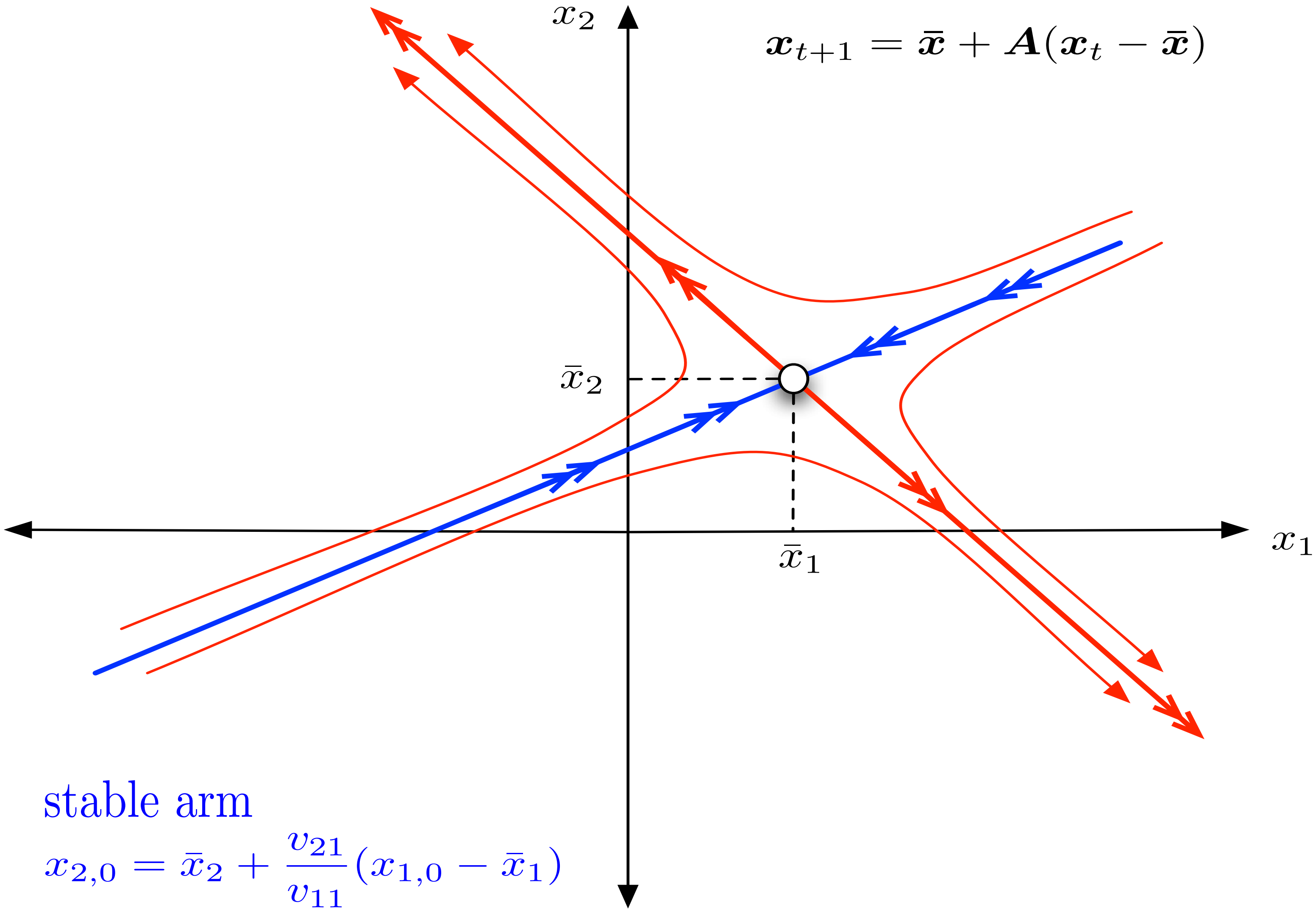
- Then system explodes except in knife-edge case $z_{2,0} = 0$

- In terms of original coordinates, a line

$$z_{2,0} = 0 \qquad \Leftrightarrow \qquad x_{2,0} = \bar{x}_2 + \frac{v_{21}}{v_{11}}(x_{1,0} - \bar{x}_1)$$

If system starts on this line ('*stable arm*','*stable manifold*') then converges to steady state. Diverges for any other initial conditions

Stable arm



stable arm

$$x_{2,0} = \bar{x}_2 + \frac{v_{21}}{v_{11}}(x_{1,0} - \bar{x}_1)$$

Some key properties of eigenvalues

- Determinant of n-by-n matrix is product of eigenvalues

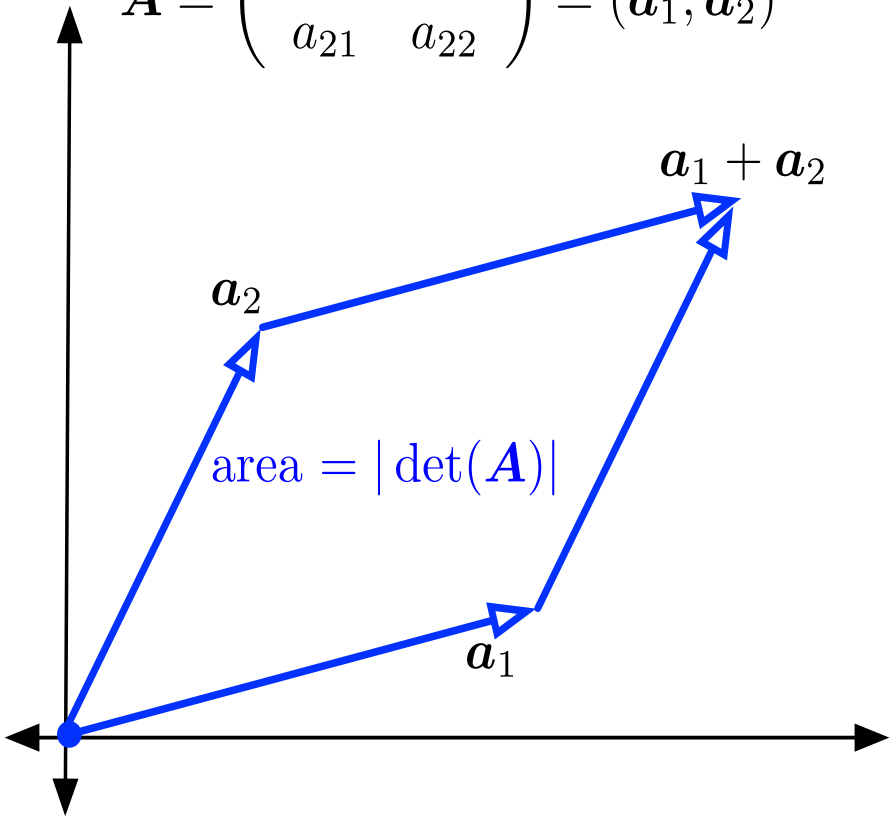
$$\det(\boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i, \qquad \det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$$

- Trace of n-by-n matrix is sum of eigenvalues

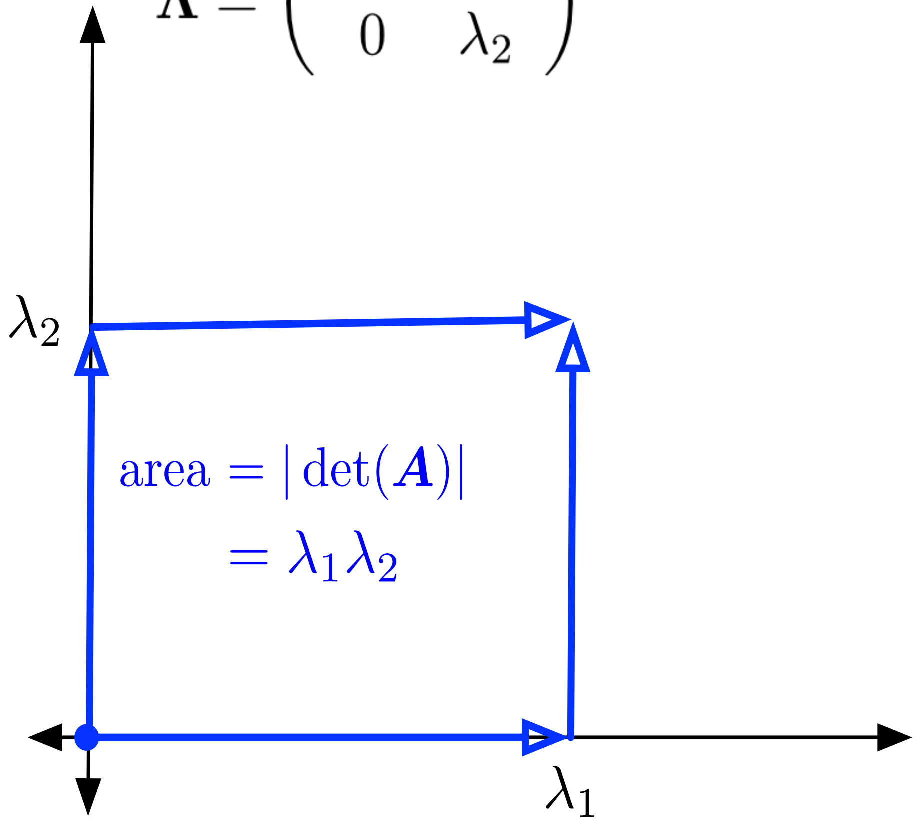
$$\mathrm{tr}(\boldsymbol{A}) \equiv \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i, \qquad \mathrm{tr}(c\boldsymbol{A}) = c\,\mathrm{tr}(\boldsymbol{A})$$

Geometric intuition

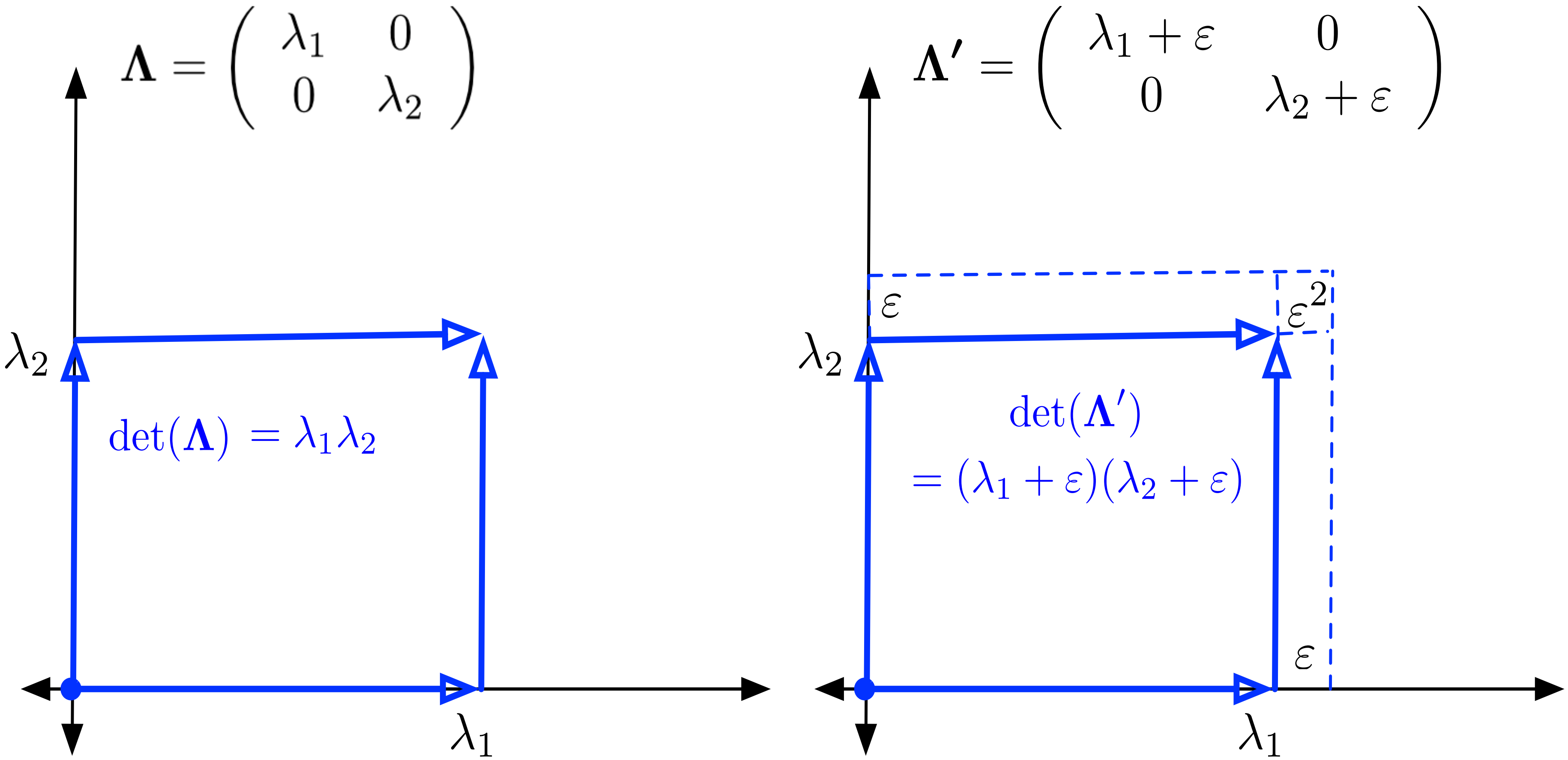
$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$$



$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$



Geometric intuition



Note $\det(\mathbf{\Lambda}') = \det(\mathbf{\Lambda}) + \mathrm{tr}(\mathbf{\Lambda})\varepsilon + \varepsilon^2$, in this sense the trace of a matrix is akin to the *derivative* of the determinant of that matrix.

Summary for 2-by-2 case

- Determinant

$$\det(\boldsymbol{A}) = a_{11}a_{22} - a_{12}a_{21} = \lambda_1\lambda_2$$

- Trace

$$\mathrm{tr}(\boldsymbol{A}) = a_{11} + a_{22} = \lambda_1 + \lambda_2$$

- Characteristic polynomial

$$p(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$= \lambda^2 - \mathrm{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A})$$

$$= (\lambda - \lambda_1)(\lambda - \lambda_2)$$

- We will use these properties to characterize magnitudes of eigenvalues and hence stability of dynamical system

Nonlinear dynamical systems

- Consider system of *nonlinear* difference equations

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} f_1(x_{1,t}, x_{2,t}) \\ f_2(x_{1,t}, x_{2,t}) \end{pmatrix}$$

or in vector notation

$$\boldsymbol{x}_{t+1} = \boldsymbol{f}(\boldsymbol{x}_t)$$

- Steady states, if any, are fixed points

$$\bar{\boldsymbol{x}} = \boldsymbol{f}(\bar{\boldsymbol{x}})$$

- Local stability of $\bar{\boldsymbol{x}}$ depends on eigenvalues of *Jacobian matrix*

$$\boldsymbol{f}'(\boldsymbol{x}) \equiv \begin{pmatrix} \frac{\partial}{\partial x_1} f_1(x_1, x_2) & \frac{\partial}{\partial x_2} f_1(x_1, x_2) \\ \frac{\partial}{\partial x_1} f_2(x_1, x_2) & \frac{\partial}{\partial x_2} f_2(x_1, x_2) \end{pmatrix}$$

evaluated at $\boldsymbol{x} = \bar{\boldsymbol{x}}$

Next class

- Application to the Ramsey-Cass-Koopmans growth model
  - a system of nonlinear difference equations
  - log-linearization (convenient local approximation)
  - solving model by method of undetermined coefficients
  - examples and introduction to Matlab