316-632 INTERNATIONAL MONETARY ECONOMICS NOTE 6c

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Relative prices in exchange economies

What follows is a slight variant of Robert Lucas's (1982) exchange economy model of international relative prices. Let there be countable dates, $t = 0, 1, 2...$ and let a state of nature be indexed by ϵ_t. A history is a vector $\epsilon^t = (\epsilon_0, \epsilon_1, ..., \epsilon_t) = (\epsilon^{t-1}, \epsilon_t)$. The unconditional probability of a history ϵ^t being realized as of date zero is denoted $\pi_t(\epsilon^t)$. The initial state ϵ_0 is known as of date zero. Let there be 2 countries with $i = 1, 2$. There are two goods that are both traded internationally. Good a only comes from country 1 while good b only comes from country 2. Each good is consumed by both countries, so there is diversification in consumption.

- *Preferences*: The representative consumer in country i has preferences over streams of consumption $c^i = \{c_t^i(\epsilon^t)\}_{t=0}^{\infty}$. These preferences are given by the expected utility function

$$\sum_{t=0}^{\infty} \sum_{\epsilon^t} \beta^t U[c_t^i(\epsilon^t)] \pi_t(\epsilon^t), \qquad 0 < \beta < 1$$

  Consumption in each country is an aggregate of the specific goods a and b, namely

$$c = G(a, b) \equiv \left[\omega a^{\frac{\gamma-1}{\gamma}} + (1-\omega) b^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}, \qquad 0 < \omega < 1 \text{ and } \gamma > 0$$

  In this CES index, the two goods become perfect substitutes as $\gamma \to \infty$ and become perfect complements as $\gamma \to 0$. The Cobb-Douglas case corresponds to $\gamma = 1$. Notice that $G(a, b)$ exhibits constant returns so that $G(a, b) = aG(1, \frac{b}{a})$ for $a > 0$. Period utility $U(c)$ is constant relative risk aversion over the aggregate quantity

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0$$

  Preferences in each country are identical.

- *Resource Constraints*: The relevant adding-up conditions are

$$\begin{aligned} a_t^1(\epsilon^t) + a_t^2(\epsilon^t) &= y_t^1(\epsilon^t) \\ b_t^1(\epsilon^t) + b_t^2(\epsilon^t) &= y_t^2(\epsilon^t) \end{aligned}$$

for each date t and ϵ^t.

- *Equivalent Planning Problem*: We will study an equivalent social planning problem. Given exogenous welfare weights $\varphi_i > 0$, the planner's problem is to maximize

$$\sum_{t=0}^{\infty}\sum_{\epsilon^t}\beta^t\pi_t(\epsilon^t)\{\varphi_1 U[c_t^1(\epsilon^t)] + \varphi_2 U[c_t^2(\epsilon^t)]\}$$

subject to the resource constraints and the definitions of the quantity indices. In what follows, define

$$V[a_t^i(\epsilon^t), b_t^i(\epsilon^t)] \equiv U\{G[a_t^i(\epsilon^t), b_t^i(\epsilon^t)]\}$$

Letting $Q_t^i(\epsilon^t) = \beta^t\pi_t(\epsilon^t)q_t^i(\epsilon^t)$ denote the Lagrange multipliers on the resource constraints, the Lagrangian is

$$\begin{aligned}\mathcal{L} \;=\; & \sum_{t=0}^{\infty}\sum_{\epsilon^t}\beta^t\pi_t(\epsilon^t)\{\varphi_1 V[a_t^1(\epsilon^t), b_t^1(\epsilon^t)] + \varphi_2 V[a_t^2(\epsilon^t), b_t^2(\epsilon^t)]\}\\ & + \sum_{t=0}^{\infty}\sum_{\epsilon^t}\beta^t\pi_t(\epsilon^t)q_t^1(\epsilon^t)[y_t^1(\epsilon^t) - a_t^1(\epsilon^t) - a_t^2(\epsilon^t)]\\ & + \sum_{t=0}^{\infty}\sum_{\epsilon^t}\beta^t\pi_t(\epsilon^t)q_t^2(\epsilon^t)[y_t^2(\epsilon^t) - b_t^1(\epsilon^t) - b_t^2(\epsilon^t)]\end{aligned}$$

Since there is no endogenous state variable (e.g., no capital), this breaks into a sequence of static problems. Suspending the event tree notation, we can write each static problem as

$$\max_{\{a_1,a_2,b_1,b_2\}} \varphi_1 V(a_1, b_1) + \varphi_2 V(a_2, b_2)$$

subject to

$$\begin{aligned} a_1 + a_2 &\leq y_1\\ b_1 + b_2 &\leq y_2\end{aligned}$$

This gives first order conditions

$$\begin{aligned}\varphi_1 V_a(a_1, b_1) &= q_1\\ \varphi_2 V_a(a_2, b_2) &= q_1\end{aligned}$$

and

$$\begin{aligned} \varphi_1 V_b(a_1, b_1) &= q_2 \\ \varphi_2 V_b(a_2, b_2) &= q_2 \end{aligned}$$

Eliminating the Lagrange multipliers to get

$$\begin{aligned} \frac{V_b(a_1, b_1)}{V_a(a_1, b_1)} &= \frac{V_b(a_2, b_2)}{V_a(a_2, b_2)} \\ \frac{V_a(a_1, b_1)}{V_a(a_2, b_2)} &= \frac{\varphi_2}{\varphi_1} \end{aligned}$$

we find that we have to solve four equations in the four unknowns a_1, a_2, b_1, b_2 (the four equations are the two just listed plus the resource constraints).

- *Solving the Model*: With the assumed utility function, namely

$$V(a, b) = \frac{1}{1-\sigma} \left\{ \left[\omega a^{\frac{\gamma-1}{\gamma}} + (1-\omega) b^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}} \right\}^{1-\sigma}$$

we can write marginal utility of a as

$$V_a(a, b) = \left\{ \left[\omega a^{\frac{\gamma-1}{\gamma}} + (1-\omega) b^{\frac{\gamma-1}{\gamma}} \right]^{\frac{1-\sigma\gamma}{\gamma-1}} \right\} \omega a^{\frac{-1}{\gamma}}$$

and similarly for the marginal utility of b

$$V_b(a, b) = \left\{ \left[\omega a^{\frac{\gamma-1}{\gamma}} + (1-\omega) b^{\frac{\gamma-1}{\gamma}} \right]^{\frac{1-\sigma\gamma}{\gamma-1}} \right\} (1-\omega) b^{\frac{-1}{\gamma}}$$

Hence

$$\frac{V_b(a_1, b_1)}{V_a(a_1, b_1)} = \frac{1-\omega}{\omega} \left(\frac{a_1}{b_1} \right)^{1/\gamma} = \frac{1-\omega}{\omega} \left(\frac{a_2}{b_2} \right)^{1/\gamma} = \frac{V_b(a_2, b_2)}{V_a(a_2, b_2)}$$

Now turning to the risk sharing condition

$$\frac{V_a(a_1, b_1)}{V_a(a_2, b_2)} = \frac{\varphi_2}{\varphi_1}$$

We can use the calculations above to write

$$\frac{\varphi_2}{\varphi_1} = \frac{\left\{\left[\omega a_1^{\frac{\gamma-1}{\gamma}} + (1-\omega) b_1^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1-\sigma\gamma}{\gamma-1}}\right\} \omega a_1^{\frac{-1}{\gamma}}}{\left\{\left[\omega a_2^{\frac{\gamma-1}{\gamma}} + (1-\omega) b_2^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1-\sigma\gamma}{\gamma-1}}\right\} \omega a_2^{\frac{-1}{\gamma}}} = \frac{\left\{\left[\omega a_1^{\frac{\gamma-1}{\gamma}} + (1-\omega) b_1^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}\right\}^{\frac{1}{\gamma}-\sigma} \omega a_1^{\frac{-1}{\gamma}}}{\left\{\left[\omega a_2^{\frac{\gamma-1}{\gamma}} + (1-\omega) b_2^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}\right\}^{\frac{1}{\gamma}-\sigma} \omega a_2^{\frac{-1}{\gamma}}}$$

$$= \frac{G(a_1, b_1)^{\frac{1}{\gamma}-\sigma}}{G(a_2, b_2)^{\frac{1}{\gamma}-\sigma}} \frac{a_1^{\frac{-1}{\gamma}}}{a_2^{\frac{-1}{\gamma}}}$$

The key trick is to notice that the ratios $\frac{a_i}{b_i}$ are constant across countries. We can combine this fact with the constant returns property of $G(a, b)$ to solve everything out. The previous calculations imply

$$\frac{\varphi_2}{\varphi_1} = \frac{[a_1 G(1, \frac{b_1}{a_1})]^{\frac{1}{\gamma}-\sigma}}{[a_2 G(1, \frac{b_2}{a_2})]^{\frac{1}{\gamma}-\sigma}} \frac{a_1^{\frac{-1}{\gamma}}}{a_2^{\frac{-1}{\gamma}}}$$

But then because $\frac{b_1}{a_1} = \frac{b_2}{a_2}$, this reduces to

$$\frac{\varphi_2}{\varphi_1} = \left(\frac{a_1}{a_2}\right)^{-\sigma}$$

or

$$\frac{a_1}{a_2} = \left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{\sigma}}$$

So we can solve for the allocations for each date t and ϵ^t

$$a_t^1(\epsilon^t) = \frac{\varphi_1^{1/\sigma}}{\varphi_1^{1/\sigma} + \varphi_2^{1/\sigma}} y_t^1(\epsilon^t), \qquad a_t^2(\epsilon^t) = \frac{\varphi_2^{1/\sigma}}{\varphi_1^{1/\sigma} + \varphi_2^{1/\sigma}} y_t^1(\epsilon^t)$$

$$b_t^1(\epsilon^t) = \frac{\varphi_1^{1/\sigma}}{\varphi_1^{1/\sigma} + \varphi_2^{1/\sigma}} y_t^2(\epsilon^t), \qquad b_t^2(\epsilon^t) = \frac{\varphi_2^{1/\sigma}}{\varphi_1^{1/\sigma} + \varphi_2^{1/\sigma}} y_t^2(\epsilon^t)$$

Each country gets a time and state independent constant fraction of the world endowment of each good. The quantities consumed of each good by each country fluctuate randomly, but only because of the random supply. Finally, solving for the Lagrange multipliers

$$\frac{q_t^2(\epsilon^t)}{q_t^1(\epsilon^t)} = \frac{V_b(a_t^1(\epsilon^t), b_t^1(\epsilon^t))}{V_a(a_t^1(\epsilon^t), b_t^1(\epsilon^t))} = \frac{1-\omega}{\omega}\left(\frac{a_t^1(\epsilon^t)}{b_t^1(\epsilon^t)}\right)^{1/\gamma} = \frac{1-\omega}{\omega}\left(\frac{y_t^1(\epsilon^t)}{y_t^2(\epsilon^t)}\right)^{1/\gamma}$$

- *International Relative Prices*: The **terms of trade** facing country 1 may be defined by

$$p_t(\epsilon^t) \equiv \frac{q_t^2(\epsilon^t)}{q_t^1(\epsilon^t)} = \frac{1-\omega}{\omega}\left(\frac{y_t^1(\epsilon^t)}{y_t^2(\epsilon^t)}\right)^{1/\gamma}$$

So the log terms of trade may be written

$$\log(p_t) = \text{constant} + \frac{1}{\gamma}\left[\log(y_t^1) - \log(y_t^2)\right]$$

and

$$\Delta\log(p_t) = \frac{1}{\gamma}\left[\Delta\log(y_t^1) - \Delta\log(y_t^2)\right]$$

Hence the moments of terms of trade movements are pinned down by the moments of output growth in the two countries and by the substitution parameter γ,

$$\begin{aligned}
\mathsf{E}\{\Delta\log(p_t)\} &= \frac{1}{\gamma}\left[\mathsf{E}\{\Delta\log(y_t^1)\} - \mathsf{E}\{\Delta\log(y_t^2)\}\right] \\
\mathsf{V}\{\Delta\log(p_t)\} &= \frac{1}{\gamma^2}\left[\mathsf{V}\{\Delta\log(y_t^1)\} + \mathsf{V}\{\Delta\log(y_t^2)\} + 2\mathsf{Cov}\{\Delta\log(y_t^1), \Delta\log(y_t^2)\}\right]
\end{aligned}$$

As $\gamma \to \infty$, the two goods become very good substitutes and the average terms of trade movement should be nearly zero with low volatility. The reverse is true as $\gamma \to 0$ so that the two goods are highly complementary; the terms of trade will be highly volatile. Because preferences are identical and because the law of one price holds in this model and both goods are traded, the price indices associated with the quantity indices c^i will be the same so the bilateral **real exchange rate** will be constant.

Relative prices in production economies

This section outlines a two country, multiple goods international RBC model that Backus, Kehoe and Kydland (1995) use to discuss international relative prices.

Again, let there be countable dates, $t = 0, 1, 2...$ and let a state of nature be indexed by ϵ_t. A history is a vector $\epsilon^t = (\epsilon_0, \epsilon_1, ..., \epsilon_t) = (\epsilon^{t-1}, \epsilon_t)$. The unconditional probability of a history ϵ^t being realized as of date zero is denoted $\pi_t(\epsilon^t)$. The initial state ϵ_0 is known as of date zero. Let there be 2 countries with $i = 1, 2$. Country 1 specializes in the production of good a while country 2 specializes in the production of good b. Each good is consumed by both countries, so there is specialization in production but diversification in consumption.

- *Preferences*: The representative consumer in country i has preferences over streams of consumption $c^i = \{c^i_t(\epsilon^t)\}_{t=0}^{\infty}$ and non-market time $\ell^i = \{\ell^i_t(\epsilon^t)\}_{t=0}^{\infty}$. These preferences are given by the expected utility function

$$\sum_{t=0}^{\infty} \sum_{\epsilon^t} \beta^t U[c^i_t(\epsilon^t), \ell^i_t(\epsilon^t)] \pi_t(\epsilon^t), \qquad 0 < \beta < 1$$

With

$$U(c, \ell) = \frac{(c^{\mu} \ell^{1-\mu})^{1-\gamma}}{1-\gamma}, \qquad 0 < \mu < 1, \gamma > 0$$

The constraint on hours worked is

$$\ell^i_t(\epsilon^t) + n^i_t(\epsilon^t) \leq 1$$

- *Resource constraints*: Good a is built in country 1 and good b is built in country 2, in each case production uses capital and labor as inputs to a constant returns to scale production function so that

$$\begin{aligned} a^1_t(\epsilon^t) + a^2_t(\epsilon^t) &\leq y^1_t(\epsilon^t) = z^1_t(\epsilon^t) F[k^1_t(\epsilon^{t-1}), n^1_t(\epsilon^t)] \\ b^1_t(\epsilon^t) + b^2_t(\epsilon^t) &\leq y^2_t(\epsilon^t) = z^2_t(\epsilon^t) F[k^2_t(\epsilon^{t-1}), n^2_t(\epsilon^t)] \end{aligned}$$

where $z^1_t(\epsilon^t)$ and $z^2_t(\epsilon^t)$ denote productivity shocks. We will typically assume the Cobb-Douglas functional form

$$F(k, n) = k^{\theta} n^{1-\theta}, \qquad 0 < \theta < 1$$

- *Armington aggregation*: Each country combines quantities of a and b into a country-specific consumption/investment good. For country 1, these composite goods are given by

$$c^1_t(\epsilon^t) + x^1_t(\epsilon^t) + g^1_t(\epsilon^t) \leq G[a^1_t(\epsilon^t), b^1_t(\epsilon^t)]$$

where $c^1_t(\epsilon^t), x^1_t(\epsilon^t)$, and $g^1_t(\epsilon^t)$ are consumption, investment, and government purchases (the latter is an exogenous shock). The function G is an aggregator, known in the literature as an **Armington aggregator**, that is constant returns to scale and is generally assumed to have the CES form

$$G(a,b) = \left[\omega a^{1-\alpha} + b^{1-\alpha}\right]^{\frac{1}{1-\alpha}}, \qquad \omega > 0 \text{ and } \alpha > 0$$

The parameter $\omega > 0$ measures the degree of home bias in domestic spending. The parameter α governs the elasticity of substitution between a and b goods, which is $\sigma = \frac{1}{\alpha}$. As $\alpha \to 0$, the goods are perfect substitutes, as $\alpha \to 1$ they have unitary (Cobb-Douglas) elasticity of substitution, and as $\alpha \to \infty$, the goods are perfect complements. Similarly, for country 2,

$$c_t^2(\epsilon^t) + x_t^2(\epsilon^t) + g_t^2(\epsilon^t) \leq G[b_t^2(\epsilon^t), a_t^2(\epsilon^t)]$$

Notice that now good b, which is the specific good of country 2, is the first argument of G.

- *Net exports*: Let $q_t^1(\epsilon^t)$ and $q_t^2(\epsilon^t)$ denote the relative prices of the a and b goods in terms of the composite good. Using the index number results from Note 4b and the specific functional form of the aggregator G, its straightforward to show that

$$c_t^1(\epsilon^t) + x_t^1(\epsilon^t) + g_t^1(\epsilon^t) = q_t^1(\epsilon^t)a_t^1(\epsilon^t) + q_t^2(\epsilon^t)b_t^1(\epsilon^t)$$

Thus we can re-write the resource constraint for country 1 as

$$\begin{aligned} y_t^1(\epsilon^t) &= a_t^1(\epsilon^t) + a_t^2(\epsilon^t) \\ &= \left[\frac{c_t^1(\epsilon^t) + x_t^1(\epsilon^t) + g_t^1(\epsilon^t)}{q_t^1(\epsilon^t)}\right] + a_t^2(\epsilon^t) - \frac{q_t^2(\epsilon^t)}{q_t^1(\epsilon^t)} b_t^1(\epsilon^t) \end{aligned}$$

The term in square brackets is **domestic absorption**, while the second term is net exports. The ratio

$$p_t(\epsilon^t) \equiv \frac{q_t^2(\epsilon^t)}{q_t^1(\epsilon^t)}$$

is the **terms of trade**, just as in the first section on the Lucas model. Following Backus et al, define the ratio of net exports to output by

$$\mathsf{nx}_t^1(\epsilon^t) = \frac{a_t^2(\epsilon^t) - p_t(\epsilon^t)b_t^1(\epsilon^t)}{y_t^1(\epsilon^t)}$$

- *Capital formation*: In each country, investment adds to the capital stock via

$$k_{t+1}^i(\epsilon^t) = (1-\delta)k_t^i(\epsilon^{t-1}) + x_t^i(\epsilon^t), \qquad 0 < \delta < 1$$

  There is no time to build.

- *Shocks*: There are two sources of uncertainty, technology shocks and government expenditure shocks. Let $z_t = \{z_t^i(\epsilon^t)\}_{i=1}^2$ and $g_t = \{g_t^i(\epsilon^t)\}_{i=1}^2$ denote vectors of realizations of the shocks. These will evolve according to vector autoregressions of the form

$$\begin{aligned} \log(z_{t+1}) &= A\log(z_t) + \varepsilon_{t+1}^z \\ \log(g_{t+1}) &= B\log(g_t) + \varepsilon_{t+1}^g \end{aligned}$$

  where A and B are fixed matrices of coefficients and the innovations ε_{t+1}^z and ε_{t+1}^g are independent and normally distributed with constant variance/covariance matrices V^z and V^g. With this notation, $\epsilon_t = (\varepsilon_t^z, \varepsilon_t^g)$.

Social planner's problem

We study the equivalent social planner's problem. For simplicity, suppose that the social planner attaches equal welfare weights 1/2 to each country.

Then the planner's problem is to choose allocations to maximize

$$\frac{1}{2}\sum_i \sum_{t=0}^{\infty} \sum_{\epsilon^t} \beta^t U[c_t^i(\epsilon^t), 1 - n_t^i(\epsilon^t)]\pi_t(\epsilon^t)$$

subject to resource constraints

$$\begin{aligned} a_t^1(\epsilon^t) + a_t^2(\epsilon^t) &= z_t^1(\epsilon^t)F[k_t^1(\epsilon^{t-1}), n_t^1(\epsilon^t)] \quad (1) \\ b_t^1(\epsilon^t) + b_t^2(\epsilon^t) &= z_t^2(\epsilon^t)F[k_t^2(\epsilon^{t-1}), n_t^2(\epsilon^t)] \quad (2) \end{aligned}$$

and the laws of motion for capital accumulation

$$\begin{aligned} c_t^1(\epsilon^t) + k_{t+1}^1(\epsilon^t) + g_t^1(\epsilon^t) &= G[a_t^1(\epsilon^t), b_t^1(\epsilon^t)] + (1-\delta)k_t^1(\epsilon^{t-1}) \quad (3) \\ c_t^2(\epsilon^t) + k_{t+1}^2(\epsilon^t) + g_t^2(\epsilon^t) &= G[b_t^2(\epsilon^t), a_t^2(\epsilon^t)] + (1-\delta)k_t^2(\epsilon^{t-1}) \quad (4) \end{aligned}$$

For each $i = 1, 2$, the planner has to choose

$$\{c_t^i(\epsilon^t), n_t^i(\epsilon^t), a_t^i(\epsilon^t), b_t^i(\epsilon^t), k_{t+1}^i(\epsilon^t)\}$$

Let the Lagrange multiplier associated with the date t and history ϵ^t resource constraint be $q_t^i(\epsilon^t)$, with i in this case indicating the specific good of country i. Similarly, let $\lambda_t^i(\epsilon^t)$ denote the multipliers on the capital accumulation equations.

Because of the asymmetric nature of the capital accumulation equations, it's helpful to write out the Lagrangian explicitly

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{2}\sum_i \sum_{t=0}^{\infty} \sum_{\epsilon^t} \beta^t U[c_t^i(\epsilon^t), 1 - n_t^i(\epsilon^t)]\pi_t(\epsilon^t) \\
& + \sum_{t=0}^{\infty} \sum_{\epsilon^t} q_t^1(\epsilon^t)\{z_t^1(\epsilon^t)F[k_t^1(\epsilon^{t-1}), n_t^1(\epsilon^t)] - a_t^1(\epsilon^t) - a_t^2(\epsilon^t)\} \\
& + \sum_{t=0}^{\infty} \sum_{\epsilon^t} q_t^2(\epsilon^t)\{z_t^2(\epsilon^t)F[k_t^2(\epsilon^{t-1}), n_t^2(\epsilon^t)] - b_t^1(\epsilon^t) - b_t^2(\epsilon^t)\} \\
& + \sum_{t=0}^{\infty} \sum_{\epsilon^t} \lambda_t^1(\epsilon^t)\{G[a_t^1(\epsilon^t), b_t^1(\epsilon^t)] + (1-\delta)k_t^1(\epsilon^{t-1}) - c_t^1(\epsilon^t) - k_{t+1}^1(\epsilon^t) - g_t^1(\epsilon^t)\} \\
& + \sum_{t=0}^{\infty} \sum_{\epsilon^t} \lambda_t^2(\epsilon^t)\{G[b_t^2(\epsilon^t), a_t^2(\epsilon^t)] + (1-\delta)k_t^2(\epsilon^{t-1}) - c_t^2(\epsilon^t) - k_{t+1}^2(\epsilon^t) - g_t^2(\epsilon^t)\}
\end{aligned}$$

For each $i = 1, 2$ and each t, ϵ^t the planner has to choose

$$\{c_t^i(\epsilon^t), n_t^i(\epsilon^t), a_t^i(\epsilon^t), b_t^i(\epsilon^t), k_{t+1}^i(\epsilon^t)\}$$

The key first order conditions include the familiar

$$\frac{\partial \mathcal{L}}{\partial c_t^i(\epsilon^t)} = 0 \Longleftrightarrow \beta^t U_{c,t}^i(\epsilon^t)\pi_t(\epsilon^t) = \lambda_t^i(\epsilon^t) \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial n_t^i(\epsilon^t)} = 0 \Longleftrightarrow \beta^t U_{\ell,t}^i(\epsilon^t)\pi_t(\epsilon^t) = q_t^i(\epsilon^t) z_t^i(\epsilon^t) F_{n,t}^i(\epsilon^t) \tag{6}$$

Notice that the marginal utility of forgone leisure is set equal to the **value** marginal product of labor. The marginal product is multiplied by $q_t^i(\epsilon^t)$, the relative price of country i's specific good. Because capital accumulation shows up both as an input in the resource constraint and also as undepreciated capital in the accumulation equation, we have

$$\frac{\partial \mathcal{L}}{\partial k^i_{t+1}(\epsilon^t)} = 0 \Longleftrightarrow \lambda^i_t(\epsilon^t) = \sum_{\epsilon'} \lambda^i_{t+1}(\epsilon^t,\epsilon')(1-\delta) + \sum_{\epsilon'} q^i_{t+1}(\epsilon^t,\epsilon') z^i_{t+1}(\epsilon^t,\epsilon') F^i_{k,t+1}(\epsilon^t,\epsilon') \tag{7}$$

We also have

$$\frac{\partial \mathcal{L}}{\partial a^1_t(\epsilon^t)} = 0 \Longleftrightarrow q^1_t(\epsilon^t) = \lambda^1_t(\epsilon^t) G^1_{1,t}(\epsilon^t) \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial a^2_t(\epsilon^t)} = 0 \Longleftrightarrow q^1_t(\epsilon^t) = \lambda^2_t(\epsilon^t) G^2_{2,t}(\epsilon^t) \tag{9}$$

and

$$\frac{\partial \mathcal{L}}{\partial b^1_t(\epsilon^t)} = 0 \Longleftrightarrow q^2_t(\epsilon^t) = \lambda^1_t(\epsilon^t) G^1_{2,t}(\epsilon^t) \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial b^2_t(\epsilon^t)} = 0 \Longleftrightarrow q^2_t(\epsilon^t) = \lambda^2_t(\epsilon^t) G^2_{1,t}(\epsilon^t) \tag{11}$$

(Notice the convention for the partial derivatives of G, I do this because a is the first argument in country 1 but is the second argument in country 2, etc). We can now use these relationships to convert between the q multipliers and the λ multipliers as required.

We will now build up the list of equations needed to fully characterize a solution.

A. Capital accumulation

Combining equations (8) and (11) with (7), we obtain Euler equations for capital accumulation in each country written solely in terms of the λ multipliers

$$\lambda^1_t(\epsilon^t) = \sum_{\epsilon'} \lambda^1_{t+1}(\epsilon^t,\epsilon')\{1-\delta + G^1_{1,t+1}(\epsilon^t,\epsilon') z^1_{t+1}(\epsilon^t,\epsilon') F^1_{k,t+1}(\epsilon^t,\epsilon')\}$$

$$\lambda^2_t(\epsilon^t) = \sum_{\epsilon'} \lambda^2_{t+1}(\epsilon^t,\epsilon')\{1-\delta + G^2_{1,t+1}(\epsilon^t,\epsilon') z^2_{t+1}(\epsilon^t,\epsilon') F^2_{k,t+1}(\epsilon^t,\epsilon')\}$$

Now using the marginal utility of consumption, as in (5), to eliminate the λ multipliers and then simplifying

$$U^1_{c,t}(\epsilon^t) = \sum_{\epsilon'} \beta U^1_{c,t+1}(\epsilon^t,\epsilon') \frac{\pi_{t+1}(\epsilon^t,\epsilon')}{\pi_t(\epsilon^t)}\{1-\delta + G^1_{1,t+1}(\epsilon^t,\epsilon') z^1_{t+1}(\epsilon^t,\epsilon') F^1_{k,t+1}(\epsilon^t,\epsilon')\} \tag{12}$$

$$U^2_{c,t}(\epsilon^t) = \sum_{\epsilon'} \beta U^2_{c,t+1}(\epsilon^t,\epsilon') \frac{\pi_{t+1}(\epsilon^t,\epsilon')}{\pi_t(\epsilon^t)}\{1-\delta + G^2_{1,t+1}(\epsilon^t,\epsilon') z^2_{t+1}(\epsilon^t,\epsilon') F^2_{k,t+1}(\epsilon^t,\epsilon')\} \tag{13}$$

Of course, we could also write this informally,

$$U^1_{c,t} = \mathsf{E}_t \left\{ \beta U^1_{c,t+1} [1 - \delta + G^1_{1,t+1} z^1_{t+1} F^1_{k,t+1}] \right\}$$

and similarly for country 2.

B. Labor supply

We can use equations (8) and (11) to eliminate the q multiplier from the marginal utility of leisure condition to obtain equations that govern labor supply

$$\frac{U^1_{\ell,t}(\epsilon^t)}{U^1_{c,t}(\epsilon^t)} \frac{1}{G^1_{1,t}(\epsilon^t)} = z^1_t(\epsilon^t) F^1_{n,t}(\epsilon^t) \quad (14)$$

$$\frac{U^2_{\ell,t}(\epsilon^t)}{U^2_{c,t}(\epsilon^t)} \frac{1}{G^2_{1,t}(\epsilon^t)} = z^2_t(\epsilon^t) F^2_{n,t}(\epsilon^t) \quad (15)$$

Why do we get these unusual terms involving derivatives of G in the optimality conditions (12)-(13) and (14)-(15)? Essentially, we need to keep converting between marginal utilities expressed in terms of the aggregate consumption good and marginal utilities expressed in terms of the country specific good. For example, in the Euler equations governing capital accumulation, the marginal cost to country 1 of forgone aggregate consumption is $U^1_{c,t}$. The marginal benefit that accrues next period depends on the random return from being able to produce more of good a, namely $z^1_{t+1} F^1_{k,t+1}$. But we then need to convert this return in terms of good a back to marginal units of the aggregate consumption good, so the relevant amount is $G^1_{1,t+1} z^1_{t+1} F^1_{k,t+1}$.

C. Risk sharing

We also have two conditions that equate marginal rates of substitution across countries. Equating (8) to (9) and equating (10) to (11) and then eliminating the λ multipliers using (5) we get

$$U^1_{c,t}(\epsilon^t) G^1_{1,t}(\epsilon^t) = U^2_{c,t}(\epsilon^t) G^2_{2,t}(\epsilon^t) \quad (16)$$

$$U^1_{c,t}(\epsilon^t) G^1_{2,t}(\epsilon^t) = U^2_{c,t}(\epsilon^t) G^2_{1,t}(\epsilon^t) \quad (17)$$

Roughly speaking, equations (12)-(13), (14)-(15), and (16)-(17) plus the resource constraints (1)-(2) and the laws of motion for capital accumulation (3)-(4) constitute a system of 10 equations in 10 unknowns, namely

$$\{c^i_t(\epsilon^t), n^i_t(\epsilon^t), a^i_t(\epsilon^t), b^i_t(\epsilon^t), k^i_{t+1}(\epsilon^t)\}$$

for each of $i = 1, 2$.

Because of the physical state variable (capital) and because of the heterogeneous preferences, this model is too difficult to solve analytically. We could approximate solutions numerically by (i) computing the non-stochastic steady state associated with $(\bar{z}^1, \bar{z}^2, \bar{g}^1, \bar{g}^2)$, then (ii) log-linearizing the model around the non-stochastic steady state, and then (iii) solving the system of difference equations by the method of undetermined coefficients. This is essentially what Backus et al do.

Once we have solved the model, we can back out the terms of trade

$$p_t(\epsilon^t) = \frac{q_t^2(\epsilon^t)}{q_t^1(\epsilon^t)} = \frac{G_{2,t}^1(\epsilon^t)}{G_{1,t}^1(\epsilon^t)} = \frac{1}{\omega}\left(\frac{a_t^1(\epsilon^t)}{b_t^1(\epsilon^t)}\right)^{\alpha}$$

and the real exchange rate will be

$$\frac{U_{c,t}^1(\epsilon^t)}{U_{c,t}^2(\epsilon^t)} = \frac{G_{2,t}^2(\epsilon^t)}{G_{1,t}^1(\epsilon^t)} = \frac{G_{1,t}^2(\epsilon^t)}{G_{2,t}^1(\epsilon^t)}$$

which in general is time and state varying.

D. Notes on price puzzles

Backus et al (1995) study quarterly data from 1970 to 1990 on 10 industrialized countries. With respect to prices, their main focus is on relative volatilities of the terms of trade and net exports. The terms of trade is defined as the ratio of an import price index to an export price index, which is the reciprocal of the usual convention but is at least consistent with the way that exchange rates are normally defined. Net exports are measured relative to output. Essentially all of their calculations refer to data detrended with the **Hodrick-Prescott filter**. Their main findings are:

- Terms of trade considerably more volatile than output, e.g, for Australia

$$\frac{\mathsf{Std}(p)}{\mathsf{Std}(y)} = \frac{5.78}{1.45} = 3.99$$

  and for the United States

$$\frac{\mathsf{Std}(p)}{\mathsf{Std}(y)} = \frac{3.68}{1.92} = 1.92$$

  Generally, the volatility of the terms of trade is higher than that of output by a factor of 2-3.

- Terms of trade considerably persistent, with quarterly autocorrelations of around 0.80.
- Terms of trade negatively correlated with net exports. United States is only significant outlier on this front. Across countries, the terms of trade is not systematically correlated with

output. It is countercyclical for Australia

$$\mathsf{Corr}(p, y) = -0.27$$

and similarly for US, but is procyclical for Italy

$$\mathsf{Corr}(p, y) = 0.38$$

In their benchmark calculations, Backus et al set $\alpha = \frac{2}{3}$ so that

$$\log(p_t) = \text{constant } + \frac{2}{3}\left[\log(a_t^1) - \log(b_t^1)\right]$$

and choose ω so that the steady state import share is 0.15. The benchmark model has about the right persistence for terms of trade movements but nowhere near the right amount of terms of trade volatility. The model also correctly gets the generally negative comovement of the terms of trade with the trade balance, but predicts a generally positive comovement of the terms of trade with output, which is not what is seen in the data. Various alternative parameterizations do not change the picture much. The main problem is that in the data, there is not nearly enough variation in the ratios $\log(\frac{a^i}{b^i})$ to explain the volatility in the observed terms of trade.

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