316-406 Advanced Macroeconomic Techniques

Problem Set #2 *Selected answers*

Question 1. The relevant plots are attached (see Figure 1) and can be produced with the Matlab code "`ps2_solutions.m`". When $a = 0.5$, the process quickly begins to fluctuate in an obviously mean-reverting manner around 0 (in the case of $b = 0$) or 2 (in the case of $b = 1$). For the relatively low value of a, increasing the sample size from $T = 100$ to $T = 500$ does not make much difference. When $a = 0.99$, the process is much more persistent. (It is not obviously mean-reverting, is it?). This is particularly a problem when $b = 1$ and we start the model at $x_0 = 0$. When $b = 1$ and $a = 0.99$, the long run mean of the process is 100 and it takes quite a while for the process to wander towards that value. If we were simulating this model and wanted only realizations from the long run distribution, we would be best advised to iterate long enough to get away from any dependence on the initial condition.

Question 2. The stationary distribution $\bar{\pi}$ is the eigenvector associated with the unit eigenvalue in the problem

$$(I - P')\bar{\pi} = 0$$

or

$$\begin{pmatrix} 1-p & -(1-p) \\ -(1-p) & 1-p \end{pmatrix} \begin{pmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Because of the linear dependency, this only gives us one equation in the two unknowns, namely

$$(1-p)\bar{\pi}_1 - (1-p)\bar{\pi}_2 = 0 \Leftrightarrow \bar{\pi}_1 = \bar{\pi}_2$$

But we also know that $\bar{\pi}_1 + \bar{\pi}_2 = 1$, so

$$\bar{\pi} = \begin{pmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Now consider the mean and variance of the stationary distribution

$$\mathsf{E}\{X_t\} = \sum_i x_i \bar{\pi}_i = (\mu + \sigma)\frac{1}{2} + (\mu - \sigma)\frac{1}{2} = \mu$$

and

$$\begin{aligned}
\mathsf{Var}\{X_t\} &= \mathsf{E}\{X_t^2\} - \mathsf{E}\{X_t\}^2 \\
&= \sum_i x_i^2 \bar{\pi}_i - \mu^2 \\
&= (\mu+\sigma)^2 \frac{1}{2} + (\mu - \sigma)^2 \frac{1}{2} - \mu^2 \\
&= \frac{1}{2}\mu^2 + \mu\sigma + \frac{1}{2}\sigma^2 + \frac{1}{2}\mu^2 - \mu\sigma + \frac{1}{2}\sigma^2 - \mu^2 \\
&= \sigma^2
\end{aligned}$$

So

$$\mathsf{Std}\{X_t\} = \sqrt{\mathsf{Var}\{X_t\}} = \sigma$$

Now let's consider the autocorrelation coefficient

$$\begin{aligned}\mathsf{Corr}\{X_t, X_{t-1}\} &= \frac{\mathsf{Cov}(X_t, X_{t-1})}{\mathsf{Std}\{X_t\}\mathsf{Std}\{X_{t-1}\}} \\ &= \frac{\mathsf{E}\{X_t X_{t-1}\} - \mathsf{E}\{X_t\}\mathsf{E}\{X_{t-1}\}}{\mathsf{Std}\{X_t\}\mathsf{Std}\{X_{t-1}\}} \\ &= \frac{\mathsf{E}\{X_t X_{t-1}\} - \mu^2}{\sigma^2}\end{aligned}$$

So all we have to do is compute the term $\mathsf{E}\{X_t X_{t-1}\}$. To do this recall that if we have two random variables X and Y, then we compute expectations by taking sums over the **joint distribution**. In this serially correlated case, we have

$$\mathsf{E}\{X_t X_{t-1}\} = \sum_i \sum_j x_i x_j \pi(x_j|x_i)\bar{\pi}_i$$

where the conditional distribution is given by the transition matrix P, that is

$$\pi(x_j|x_i) = \Pr(X_{t+1} = x_j | X_t = x_i) = P_{ij}$$

So

$$\begin{aligned}\mathsf{E}\{X_t X_{t-1}\} &= x_1 x_1 P_{11}\bar{\pi}_1 + x_1 x_2 P_{12}\bar{\pi}_1 + x_2 x_1 P_{21}\bar{\pi}_2 + x_2 x_2 P_{22}\bar{\pi}_2 \\ &= (\mu+\sigma)^2\frac{p}{2} + 2(\mu+\sigma)(\mu-\sigma)\frac{1-p}{2} + (\mu-\sigma)^2\frac{p}{2} \\ &= \left(\mu^2 + 2\mu\sigma + \sigma^2\right)\frac{p}{2} + (\mu^2 - \sigma^2)(1-p) + \left(\mu^2 - 2\mu\sigma + \sigma^2\right)\frac{p}{2} \\ &= p\mu^2 + p\sigma^2 + (1-p)\mu^2 - (1-p)\sigma^2 \\ &= \mu^2 + (2p-1)\sigma^2\end{aligned}$$

Putting this together with our previous calculation

$$\mathsf{Corr}\{X_t, X_{t-1}\} = \frac{\mathsf{E}\{X_t X_{t-1}\} - \mu^2}{\sigma^2} = 2p - 1$$

So, when $p = \frac{1}{2}$, the serial correlation is zero and the Markov chain is IID. When $p \to 1$, there is perfect positive serial correlation. When $p \to 0$, then there is perfect negative serial correlation.

The code needed to run the simulations is in "ps2_solutions.m". In my runs, this gives rise to the table

	$T = 50$	$T = 100$	$T = 1000$
	$\bar{x} = 0.0216$	$\bar{x} = 0.0200$	$\bar{x} = 0.0206$
$p = 0.1$	$s = 0.0404$	$s = 0.0402$	$s = 0.0400$
	$\rho = -0.8397$	$\rho = -0.8788$	$\rho = -0.7782$
	$\bar{x} = 0.0280$	$\bar{x} = 0.0168$	$\bar{x} = 0.0195$
$p = 0.5$	$s = 0.0396$	$s = 0.0401$	$s = 0.0400$
	$\rho = 0.0209$	$\rho = -0.1183$	$\rho = -0.0052$
	$\bar{x} = 0.0328$	$\bar{x} = 0.0160$	$\bar{x} = 0.0193$
$p = 0.9$	$s = 0.0383$	$s = 0.0400$	$s = 0.0400$
	$\rho = 0.7725$	$\rho = 0.7348$	$\rho = 0.8178$

Recall that in each case the population moments are $\mu = 0.02$ and $\sigma = 0.04$ with autocorrelation $= -0.8, 0.0, 0.8$ depending on the value of p. Generally, we see that the larger the sample size the closer the sample statistics to their population counterparts.

Question 3. Let

$$u_i = \frac{c_i^{1-\gamma}}{1-\gamma}$$

Then the vector v is found from

$$v = \sum_{t=0}^{\infty} \beta^t P^t u = \sum_{t=0}^{\infty} (\beta P)^t u$$

Because $0 < \beta < 1$ and P is a stochastic matrix, the sequence $(\beta P)^t$ is convergent and we can write

$$v = (I - \beta P)^{-1} u$$

which is the natural matrix analogue of the usual formula for a geometric series.

Until the initial state is known, v is random with expected value

$$V = \mathsf{E}\{v\} = \sum_i v_i \pi_{0,i}$$

Now let

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

With $\gamma = 2.5$, the expected payoffs are

$$\begin{aligned} V_{(P_1)}^{(2.5)} &= -7.2630 \\ V_{(P_2)}^{(2.5)} &= -7.2630 \end{aligned}$$

So the consumer is indifferent, despite the fact that P_1 is entirely persistent while P_2 is IID. Why? Notice that the consumer does not know which state she will begin in and that she is assigned to the initial state with half/half probability. Therefore under P_1 she has a 50% probability of being stuck with consumption $c = c_L = 1$ and a 50% chance of getting $c = c_H = 5$ forever. On the other hand, under P_2 she is not stuck and spends half her time in the high state and half her time in the low state. So ex ante she is indifferent between the two chains. This reasoning does not depend on the level of risk aversion, so we are not surprised that she is still indifferent when $\gamma = 4.0$, that is

$$\begin{aligned} V_{(P_1)}^{(4.0)} &= -3.3600 \\ V_{(P_2)}^{(4.0)} &= -3.3600 \end{aligned}$$

Alternatively, if the initial distribution gave a high probability to the high state, then obviously she would prefer the chain with transitions P_1, and similarly she would prefer the chain with transitions P_2 if the initial distribution gave high probability to the low state.

Question 4. The "big" Markov chain has four states

$$x = (x_{HH}, x_{HL}, x_{LH}, x_{LL})$$

where

$$\begin{aligned} x_{HH} &= (g_H, e_H) \\ x_{HL} &= (g_H, e_L) \\ x_{LH} &= (g_L, e_H) \\ x_{LL} &= (g_L, e_L) \end{aligned}$$

The transition matrix for this chain has the structure

$$\pi(x_j|x_i) = \Pr(X_{t+1} = x_j|X_t = x_i) = P_{ij}$$

$$P_x = \begin{pmatrix} p_{HH,HH} & p_{HL,HH} & p_{LH,HH} & p_{LL,HH} \\ p_{HH,HL} & p_{HL,HL} & p_{LH,HL} & p_{LL,HL} \\ p_{HH,LH} & p_{HL,LH} & p_{LH,LH} & p_{LL,LH} \\ p_{HH,LL} & p_{HL,LL} & p_{LH,LL} & p_{LL,LL} \end{pmatrix}$$

where, for example,

$$\begin{aligned} p_{HL,HH} &= \Pr(X_{t+1} = x_{HL}|X_t = x_{HH}) \\ &= \Pr\left(\begin{array}{c|c} G_{t+1} = g_H & G_t = g_H \\ E_{t+1} = e_L & E_t = e_H \end{array} \right) \\ &= \Pr\left(E_{t+1} = e_L|E_t = e_H, G_t = g_H\right) \times \Pr(G_{t+1} = g_H|G_t = g_H) \\ &= p_{12}^H q_{11} \end{aligned}$$

where p_{12}^H is the $1,2$ element of the matrix P_H. Similar reasoning leads to

$$P_x = \begin{pmatrix} p_{11}^H q_{11} & p_{12}^H q_{11} & p_{11}^L q_{12} & p_{12}^L q_{12} \\ p_{21}^H q_{11} & p_{22}^H q_{11} & p_{21}^L q_{12} & p_{22}^L q_{12} \\ p_{11}^H q_{21} & p_{12}^H q_{21} & p_{11}^L q_{22} & p_{12}^L q_{22} \\ p_{21}^H q_{21} & p_{22}^H q_{21} & p_{21}^L q_{22} & p_{22}^L q_{22} \end{pmatrix}$$

Notice that this is a bona-fide transition matrix so that, for example,

$$\begin{aligned} p_{11}^H q_{11} + p_{12}^H q_{11} + p_{11}^L q_{12} + p_{12}^L q_{12} &= p_{11}^H q_{11} + (1 - p_{11}^H) q_{11} + p_{11}^L (1 - q_{11}) + (1 - p_{11}^L)(1 - q_{11}) \\ &= 1 \end{aligned}$$

In Matlab we can easily set up the transition matrix P_x by writing it as

$$P_x = \begin{pmatrix} q_{11} P_H & q_{12} P_L \\ q_{21} P_H & q_{22} P_L \end{pmatrix}$$

To complete the description of the Markov chain, we have the initial distribution

$$\pi_0^x = (1, 0, 0, 0)$$

(since we know that we start for sure in state $x_{HH} = (g_H, e_H)$.

For the transition matrices

$$Q = \begin{pmatrix} 0.99 & 0.01 \\ 0.25 & 0.75 \end{pmatrix}, \qquad P_H = \begin{pmatrix} 0.99 & 0.01 \\ 0.9 & 0.1 \end{pmatrix}, \qquad P_L = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}$$

the attached Matlab code computes the invariant distribution

$$\begin{aligned} \bar{\pi}^x &= (\bar{\pi}^x_{HH}, \bar{\pi}^x_{HL}, \bar{\pi}^x_{LH}, \bar{\pi}^x_{LL}) \\ &= (0.9503, 0.0112, 0.0109, 0.0275) \end{aligned}$$

The economy is in recession with probability $0.0109 + 0.0275 = 0.0384$ or 3.85% of the time while an individual is unemployed with probability $0.0112 + 0.0275 = 0.0387$ or 3.87% of the time. A figure shows some sample realizations from this Markov chain.

For the transition matrices

$$Q = \begin{pmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{pmatrix}, \qquad P_H = \begin{pmatrix} 0.99 & 0.01 \\ 0.99 & 0.01 \end{pmatrix}, \qquad P_L = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

the invariant distribution is

$$\begin{aligned} \bar{\pi}^x &= (\bar{\pi}^x_{HH}, \bar{\pi}^x_{HL}, \bar{\pi}^x_{LH}, \bar{\pi}^x_{LL}) \\ &= (0.495, 0.005, 0.25, 0.25) \end{aligned}$$

The economy is in recession with probability $0.25 + 0.25 = 0.50$ or 50% of the time while an individual is unemployed with probability $0.005 + 0.25 = 0.255$ or 2.55% of the time.

A figure shows some sample realizations from this Markov chain. For clarity, I made the samples of length $T = 250$ in this case. As you can see, when the economy is in recession, the conditional volatility of employment status is much higher. This second example has the same properties as the first, but I made the transition properties higher so as to emphasize the state-dependent nature of the conditional volatility.

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