316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 7a

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Introduction to incomplete markets

We will now turn our attention to models without a representative agent. In particular, we will study a class of models — sometimes known as "Bewley" models — where there is a large cross-section of agents who experience idiosyncratic risk that they cannot trade away through perfect insurance markets.

The first model that we will study is due to Mark Huggett (1993, JEDC). Huggett is interested in a particular issue, the "risk free rate puzzle" but his paper is also interesting for its methodological contribution.

A. Risk free rate puzzle

To understand the risk free rate puzzle, consider the simplest possible response to the equity premium: maybe people really are really *really* risk averse. For example, maybe people have coefficients of relative risk aversion of $\sigma = 20$. To see why this is not a satisfactory answer, notice that the long run average risk free interest rate will satisfy a steady-state condition like

$$(1+g)^{\sigma} = \beta(1+r)$$

where g is the long run average consumption growth rate, β is the time discount factor, and r is the risk free rate. Then we can write

$$r \simeq -\log(\beta) + \sigma g$$

So if you crank up risk aversion, you must also crank up the risk free rate. To get some sense of the quantitative importance of this, recall that in Mehra and Prescott's data, consumption growth is about $g = 0.018$ on an annual basis. A typical estimate of β is about $\beta = 0.96$ on an annual basis, so the pure rate of time preference is something like $-\log(\beta) = 0.04$. Suppose then that agents were risk neutral ($\sigma = 0$). Then the risk free rate would merely reflect the rate of time preference and $r = 0.04$ or 4% on an annual basis. If agents had log preferences ($\sigma = 1$), then the risk free rate would be $r = 0.058$, about 6% on an annual basis. And if agents are really *really* risk averse, say $\sigma = 20$, then the risk free rate would be $r = 0.40$, about 40% on an annual basis! This is too high. If we crank up risk aversion to explain the equity premium we simply create a new puzzle in that we also make the risk free rate too big.

B. Huggett's model

Huggett proposes a heterogeneous agent model with incomplete markets. He studies a world where the only uncertainty is **idiosyncratic endowment risk**. There is no aggregate uncertainty (e.g., no business cycle). If a complete set of state-contingent claims could be traded, then agents would be able to perfectly insure themselves against their idiosyncratic risk. But in Huggett's model, markets are incomplete and so this insurance cannot be achieved. Instead, household's **self-insure** by acquiring a "buffer-stock" of savings to help them smooth out idiosyncratic shocks. But if household's tend to run up large quantities of savings, the risk free real interest rate will be relatively low (compared to a representative agent or complete markets benchmark). This may help us explain the risk free rate puzzle.

In particular, Huggett considers a model where the **only** asset that can be traded is a one-period un-contingent bond — i.e., a claim to one unit of consumption for sure to be delivered in one period's time. Moreover, Huggett imposes a borrowing constraint that prevents households issuing too much debt. Finally, his is a model where there are no outside assets so that the net bond position across all households must be zero. At any point in time, some households can be lenders and some can be borrowers (but not too much), so long as the average bond position is zero. The price of the bond, denoted q, has to be such that the bond market clears. The risk free rate r is defined by $q = 1/(1+r)$. Unlike Mehra and Prescott's model, there is no trend in consumption so the relevant representative agent "benchmark" is just $r = \frac{1-\beta}{\beta} \simeq -\log(\beta)$.

An individual's endowment y follows a Markov chain with transition probabilities $\pi(y', y)$ where

$$\pi(y', y) = \Pr(y_{t+1} = y' | y_t = y)$$

and where $y, y' \in \mathcal{Y}$, a finite set. Taking as given the bond price q, a typical household has the dynamic programming problem

$$V(a, y) = \max_{a' \geq \underline{a}} \left\{ U(c) + \beta \sum_{y'} V(a', y') \pi(y', y) \right\}$$

where the maximization on the right hand side is subject to the budget constraint

$$c + qa' \leq a + y$$

Here, a for assets denotes the stock of bonds brought into the current period (which pay one unit of consumption each) and a' denotes the stock of bonds taken into the subsequent period. The choice of bonds is restricted by the borrowing constraint $a' \geq \underline{a}$ where $\underline{a} < 0$ is a parameter of the model. For example, Huggett often uses a borrowing limit of $\underline{a}$ equal to one year's average income.

A solution to this dynamic programming problem is a value function $V(a, y)$ and an individual decision rule $a' = g(a, y)$. Both of these decision rules also depend on the price q, so I sometimes also write $V(a, y, q)$ and $g(a, y, q)$.

When we compute an equilibrium of this model, we will restrict bond choices to belong to a discrete grid of values, say

$$a' \in \mathcal{A} \equiv [\underline{a} < \cdots < \overline{a}]$$

where $\underline{a} < 0$ is the borrowing limit, which may be binding, and $\overline{a} > 0$ is a non-binding upper limit on the grid of possible bond choices. We can then identify a household's individual state (a, y) with a point in $\mathcal{A} \times \mathcal{Y}$. We describe the cross-section of households by the (probability) **distribution** of household's over the state space $\mathcal{A} \times \mathcal{Y}$. The only ways in which households differ is that they may have either a different idiosyncratic endowment realization, y, or different bond holdings, a. Write this distribution as

$$\mu_t(a, y) \equiv \Pr(a_t = a, y_t = y)$$

This distribution has two interpretations. First, for a given household $\mu_t(a, y)$ tells us the likelihood that at date t that given household has state (a, y). Alternatively, $\mu_t(a, y)$ tells us the population fraction of all households that have the state (a, y) at t. So $\mu_t(a, y)$ tells us **time-series** properties for an individual household as well as **cross-section** properties for the population of households. Also, because the probability distribution $\mu_t(a, y)$ is "induced" from the exogenous Markov chain for y and the policy function $g(a, y, q)$, we might also want to acknowledge the implicit dependence on q by writing $\mu_t(a, y, q)$.

C. Equilibrium concept

Now notice that given the way I have written down this household's decision problem, the bond price q is constant. This is restrictive. In general, the bond price q would fluctuate as a function of the aggregate state of the economy. And what is the aggregate state of the economy? It is the distribution of state variables across households, μ_t. In order to justify a constant bond price, we will study a more restrictive equilibrium concept where the distribution of state variables is itself constant over time, say μ. If so, the bond price will also be constant.

DEFINITION. A **stationary recursive competitive equilibrium** for this economy is: (i) a value function V, (ii) an individual decision rule g, (iii) a stationary probability distribution μ, and (iv) a bond price q such that:

1. Given the bond price q, the value function V and the individual decision rule g solve the household's dynamic programming problem,
2. The stationary distribution μ is induced via the exogenous Markov chain for y, y' and the policy function g, and
3. The bond market clears
$$\sum_a \sum_y g(a,y,q)\mu(a,y,q) = 0$$

The first condition is fairly straightforward. If we set q, we could solve the household's dynamic programming problem by value function iteration to obtain $V(a,y,q)$ and $g(a,y,q)$. The policy function $a' = g(a,y,q)$ together with the exogenous endowment process for y, y' then induces a Markov chain on the state space $\mathcal{A} \times \mathcal{Y}$. This new Markov chain has transition probabilities from (a,y) to (a',y') given by the formula

$$\Pr(a_{t+1} = a', y_{t+1} = y' | a_t = a, y_t = y) = \Pr(a_{t+1} = a' | a_t = a, y_t = y) \Pr(y_{t+1} = y' | y_t = y)$$

But the probability $\Pr(a_{t+1} = a' | a_t = a, y_t = y)$ is either 1 if $a' = g(a,y,q)$ or 0 otherwise. So if we write an indicator function

$$I_g(a', a, y, q) \equiv \begin{cases} 1, & \text{if } a' = g(a,y,q) \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

then we can compute the transition probabilities on $\mathcal{A} \times \mathcal{Y}$ using

$$\Pr(a_{t+1} = a', y_{t+1} = y' | a_t = a, y_t = y) = I_g(a', a, y, q)\pi(y', y)$$

(I use the subscript g to emphasize the dependence on the policy function). The stationary distribution $\mu(a,y,q)$ is now found by solving for the eigenvector associated with a unit eigenvalue of the transition matrix on $\mathcal{A} \times \mathcal{Y}$.

Given the solution for $\mu(a,y,q)$, one computes the stationary aggregate demand for bonds, namely

$$\sum_a \sum_y g(a,y,q)\mu(a,y,q)$$

If this aggregate demand is positive, adjust the bond price q up until the market clears. Else, if the aggregate demand is negative, reduce the bond price q until the market clears.

In practice, this means we have an algorithm of the following kind:

1. Guess an initial bond price q_0.
2. Using that guess, solve the household's dynamic programming problem to obtain the policy function $g_0(a,y)$.
3. Using that policy function and the Markov chain for endowments, construct the transition matrix on the state space $\mathcal{A} \times \mathcal{Y}$ and solve for the stationary probability distribution $\mu_0(a,y)$ over $\mathcal{A} \times \mathcal{Y}$.
4. Compute the aggregate demand for bonds $\sum_a \sum_y g_0(a,y)\mu_0(a,y)$. If this is positive, try again with a new guess $q_1 > q_0$. If the aggregate demand for bonds is negative, try again with a new guess $q_1 < q_0$.
5. Keep iterating on $q_k \to q_{k+1}$, for $k = 0, 1, ...$, until some pre-defined convergence criterion has been met.

Notice that in an equilibrium of this kind, individual bond holdings and consumption are **stochastic processes** given by

$$\begin{aligned} a_{t+1} &= g(a_t, y_t) \\ c_t &= a_t + y_t - q a_{t+1} \end{aligned}$$

where y_t follows a Markov chain. Individual bond holdings and consumption fluctuate, but the distribution of such bond and consumption positions across households is constant. At any date, the **fraction** of households with asset position a and endowment y is constant and given by $\mu(a,y)$.

D. Calibration

Huggett works with a 2-state Markov chain for endowments, $\mathcal{Y} = \{y_L, y_H\}$ where $y_L < y_H$ are interpreted as earnings when unemployed and employed, respectively. He calibrates the Markov chain for endowments to data on the variability of earnings and average unemployment duration.

This leads to

$$\begin{aligned} y_L &= 0.1 \\ y_H &= 1.0 \end{aligned}$$

with transition matrix given by

$$\begin{aligned} \pi(y_H, y_H) &= 0.925 \\ \pi(y_H, y_L) &= 0.500 \end{aligned}$$

If there are six time periods per year, then the coefficient of variation for earnings with these parameters is 0.20 while the average unemployment spell is two periods (or $52 \times \frac{2}{6} = 17\frac{1}{3}$ weeks). The discount factor is 0.96 on an annual basis or a per period discount factor of

$$\beta = 0.96^{1/6} = 0.9932$$

Huggett also assumes constant relative risk aversion period utility,

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

with coefficient $\sigma = 1.5$, which is a conservative number. He experiments with various borrowing constraints $\underline{a}$, but his preferred number is $\underline{a} = -5.3$, which is allows borrowing of up to a year's average endowment.

E. Results

The following table is taken from Huggett's paper:

Table 1

Borrowing constraint, $\underline{a}$	Annual risk free rate	Period bond price, $q = \frac{1}{1+r}$
-2	-7.1%	1.0124
-4	2.3%	0.9962
-6	3.4%	0.9944
-8	4.0%	0.9935

Basically, as the borrowing constraint falls ($\underline{a}$ becomes more negative), the constraint becomes less and less binding and the annual risk free rate and bond price approach the representative agent benchmarks of 4.0% and $q = \beta$ respectively. For "tighter" borrowing constraints, the risk free rate is lower and the price of a bond is higher. The risk free rate is lower because it must be consistent with the large "buffer stock" of savings that households acquire to self-insure.

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